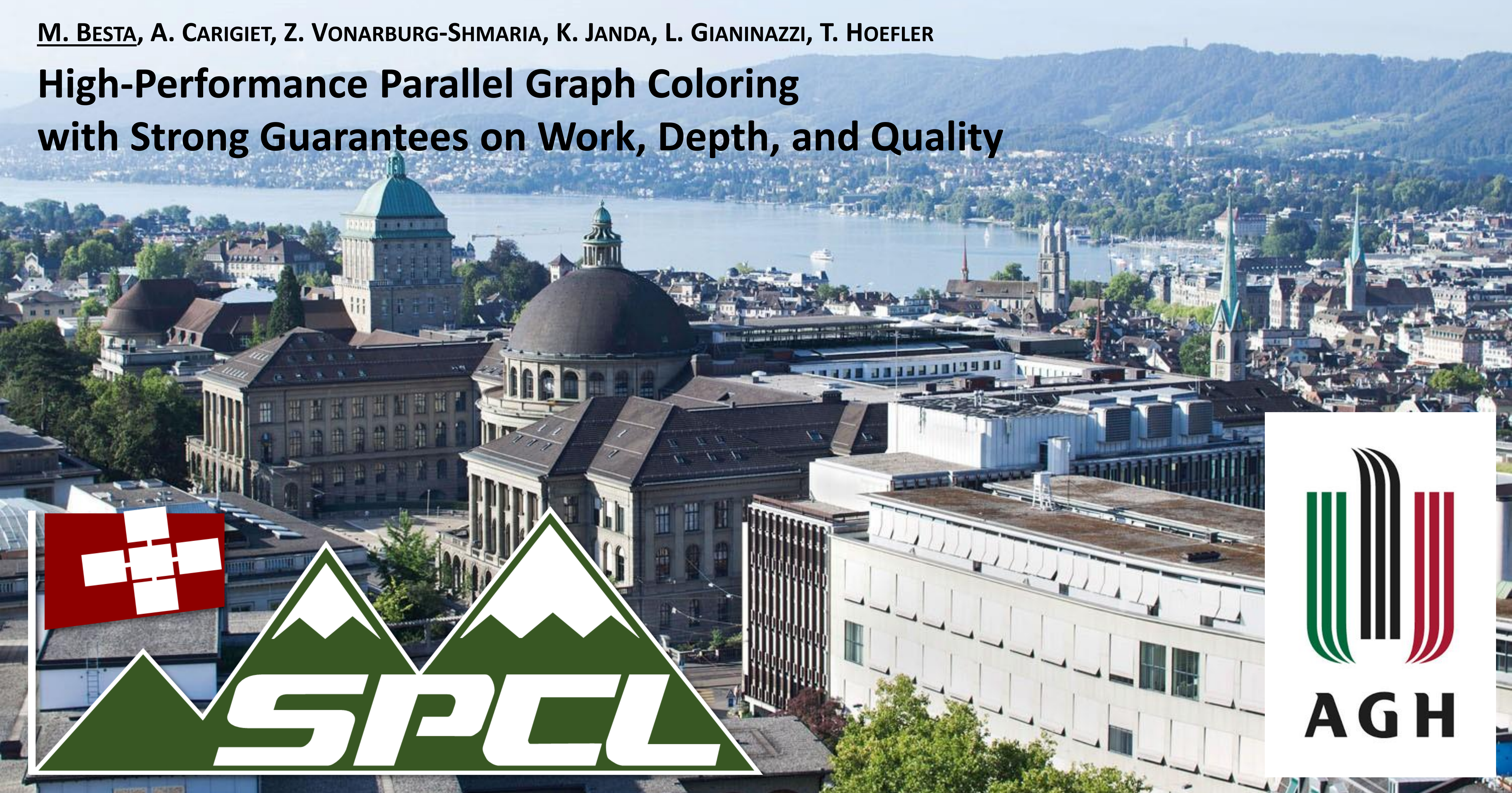
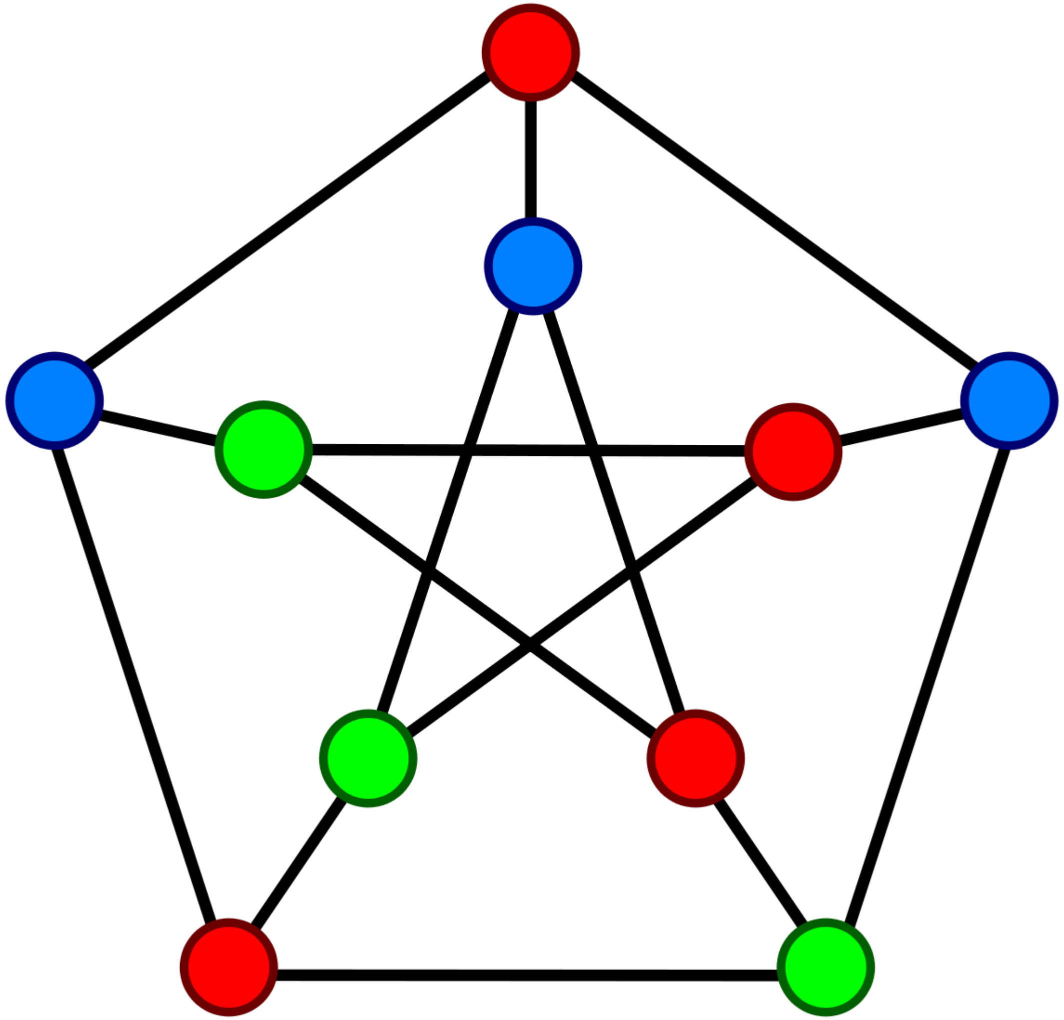
M. Besta, A. Carigiet, Z. Vonarburg-Shmaria, K. Janda, L. Gianinazzi, T. Hoefler

High-Performance Parallel Graph Coloring with Strong Guarantees on Work, Depth, and Quality

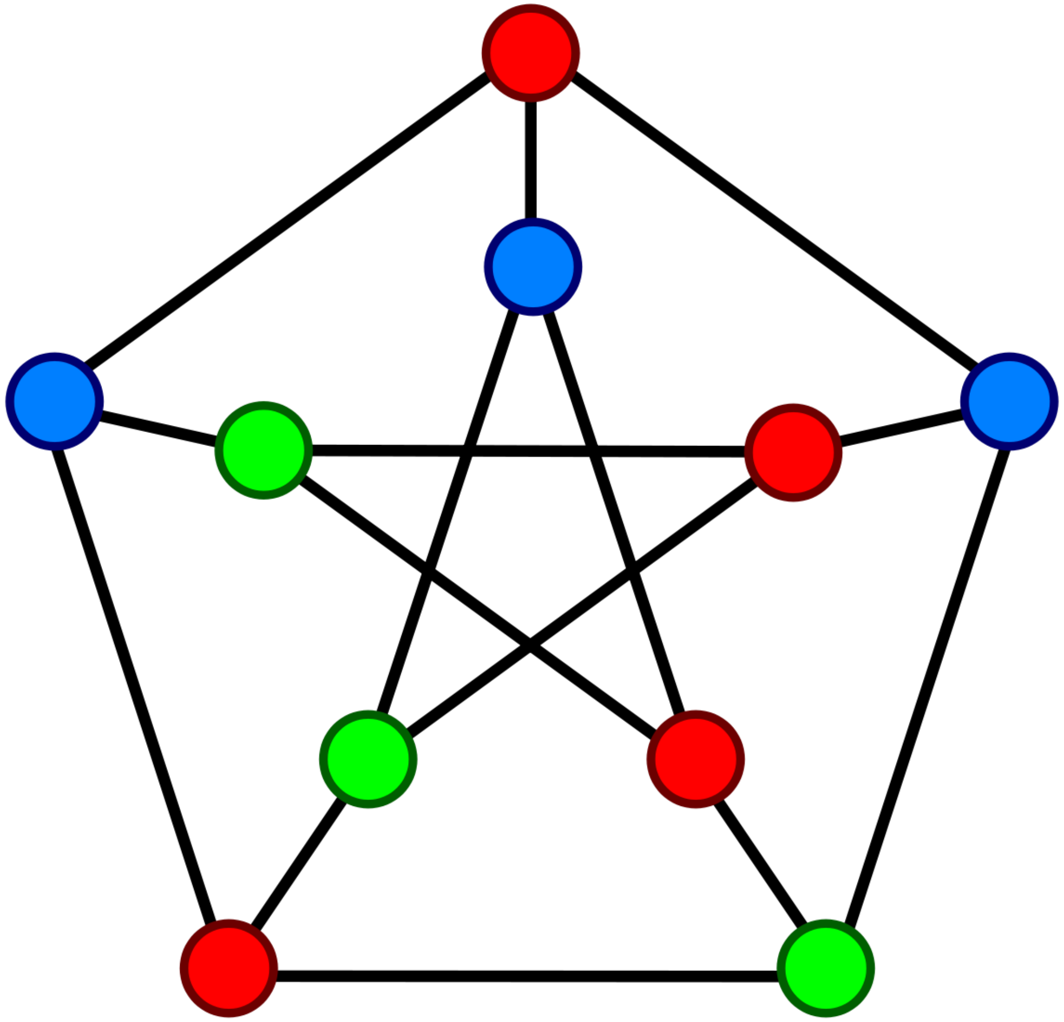


Graph coloring



Graph coloring

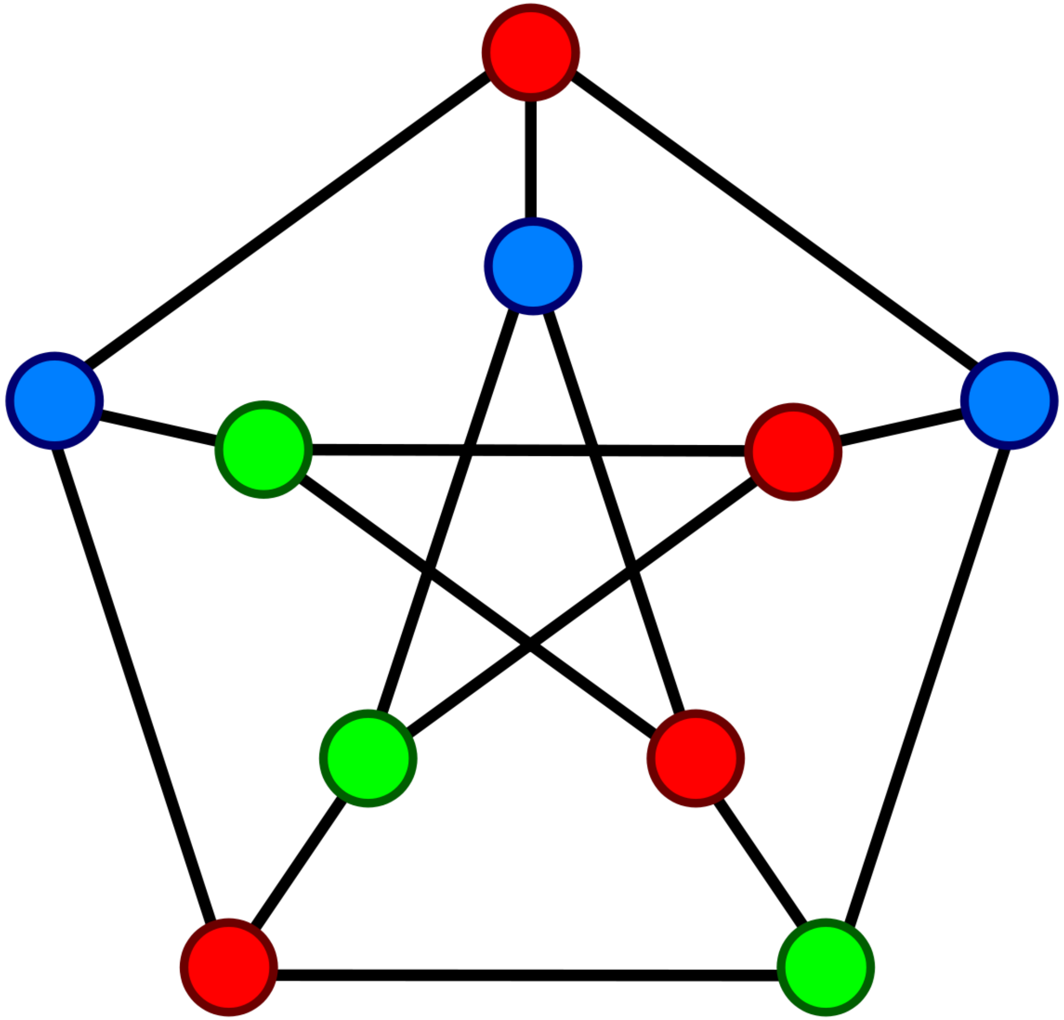
Fundamental graph problem



Graph coloring

Fundamental graph problem

Assign numbers, i.e., **colors**, to each vertex, such that **no adjacent vertices have the same color.**

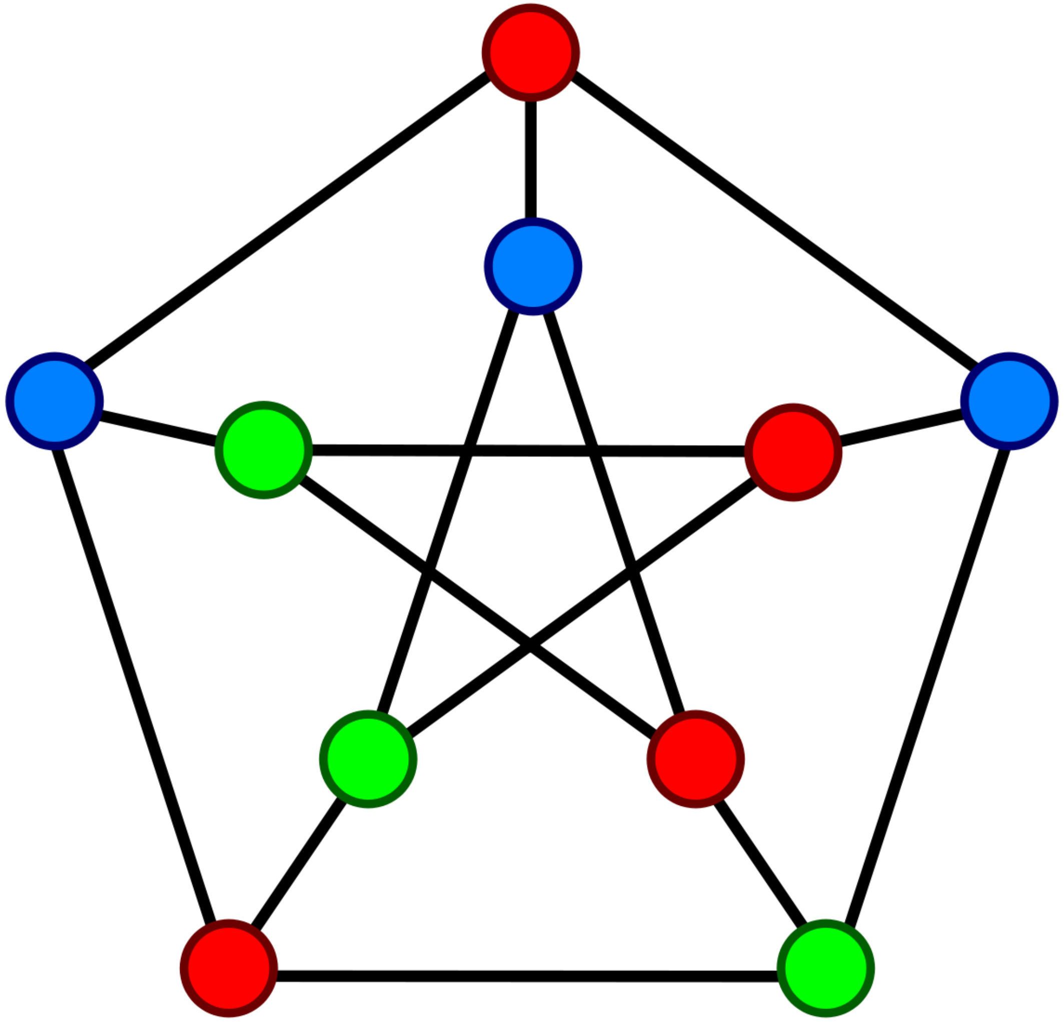


Graph coloring

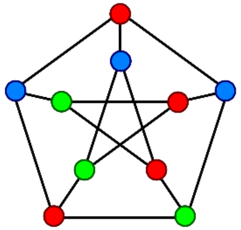
Fundamental graph problem

Assign numbers, i.e., **colors**, to each vertex, such that **no adjacent vertices have the same color.**

Goal: minimize the number of used colors



Graph coloring: applications



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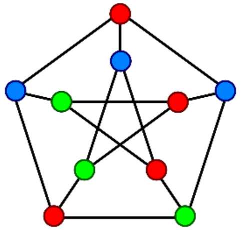

Constructing a schedule or a time-table

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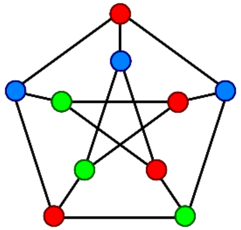
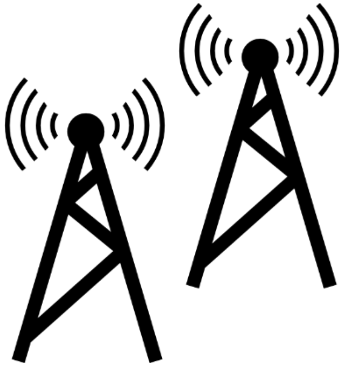


Assigning frequencies to radio towers



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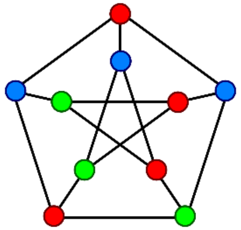

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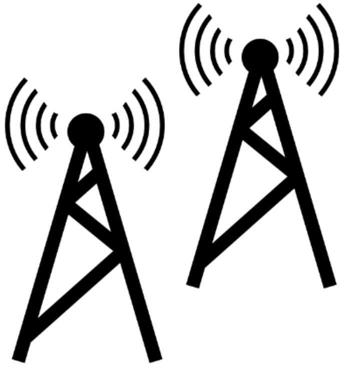

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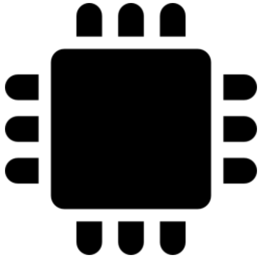

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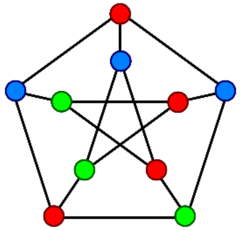

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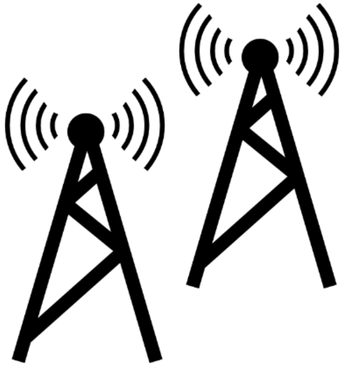

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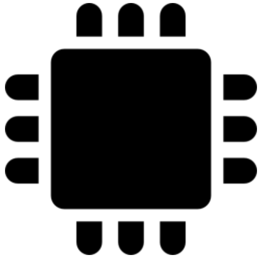

Coloring maps

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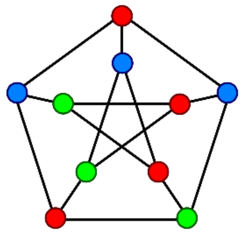

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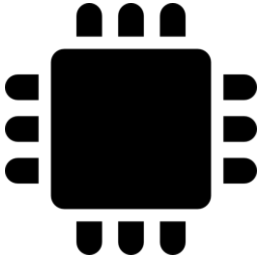
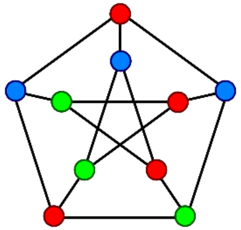


Coloring maps

Solving Sudoku ☺

Graph coloring & today's graph computations



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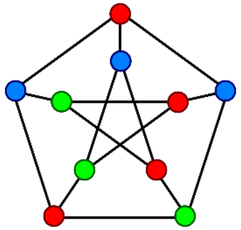

Graph datasets are **huge**

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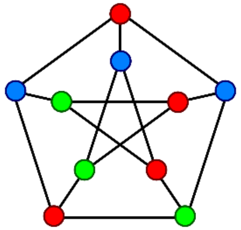

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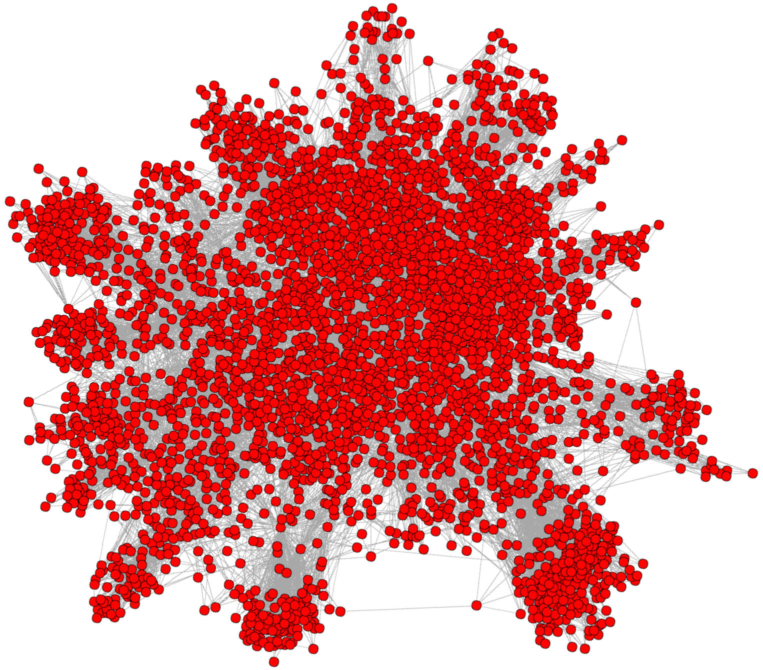
[1] Heng Lin et al.: ShenTu: Processing Multi-Trillion Edge Graphs on Millions of Cores in Seconds, **SC18, Gordon Bell Finalist**

> 233 TB
271 billion vertices,
12 trillion edges [1]



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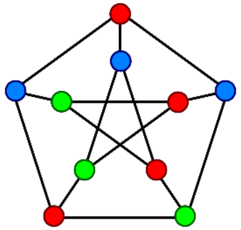

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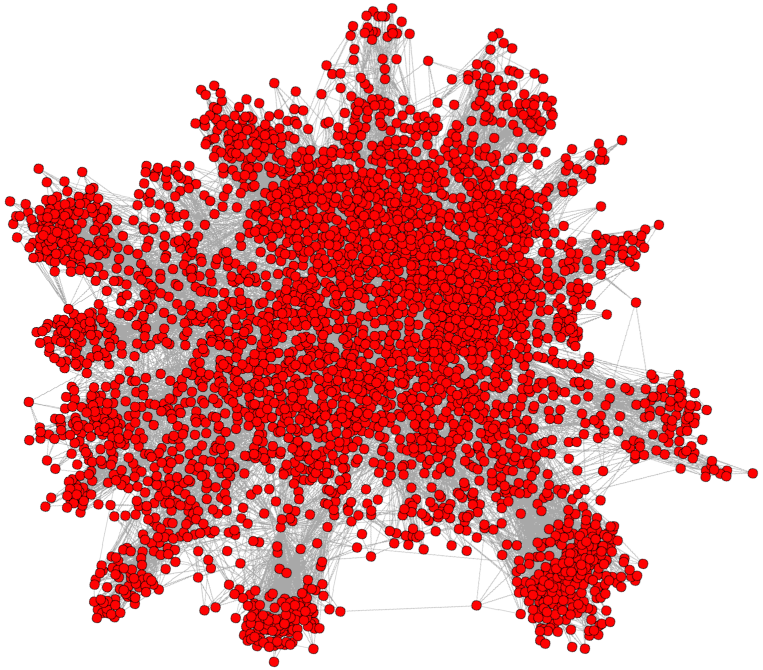

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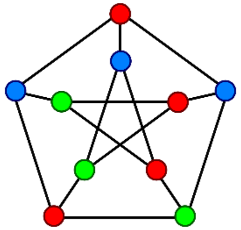

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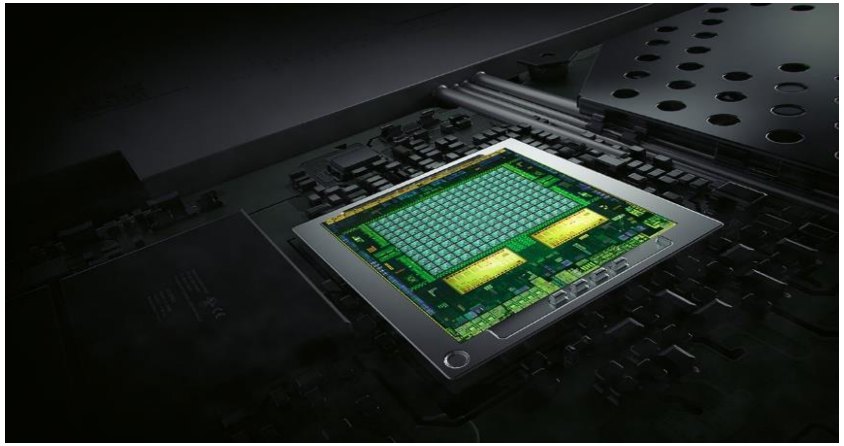

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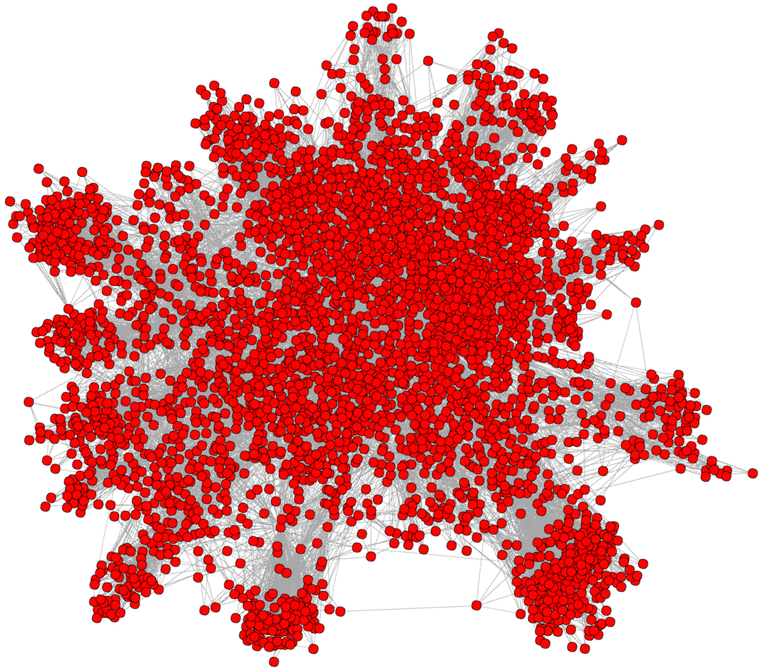

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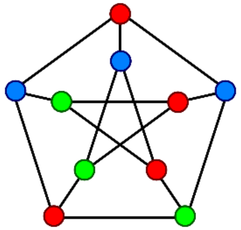

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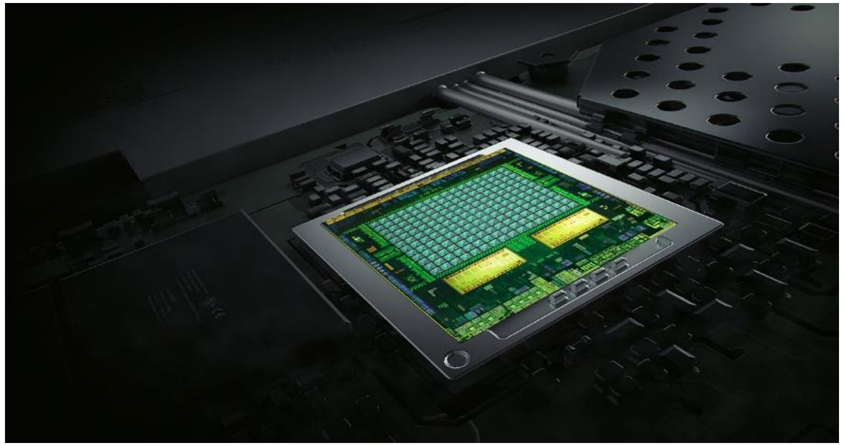

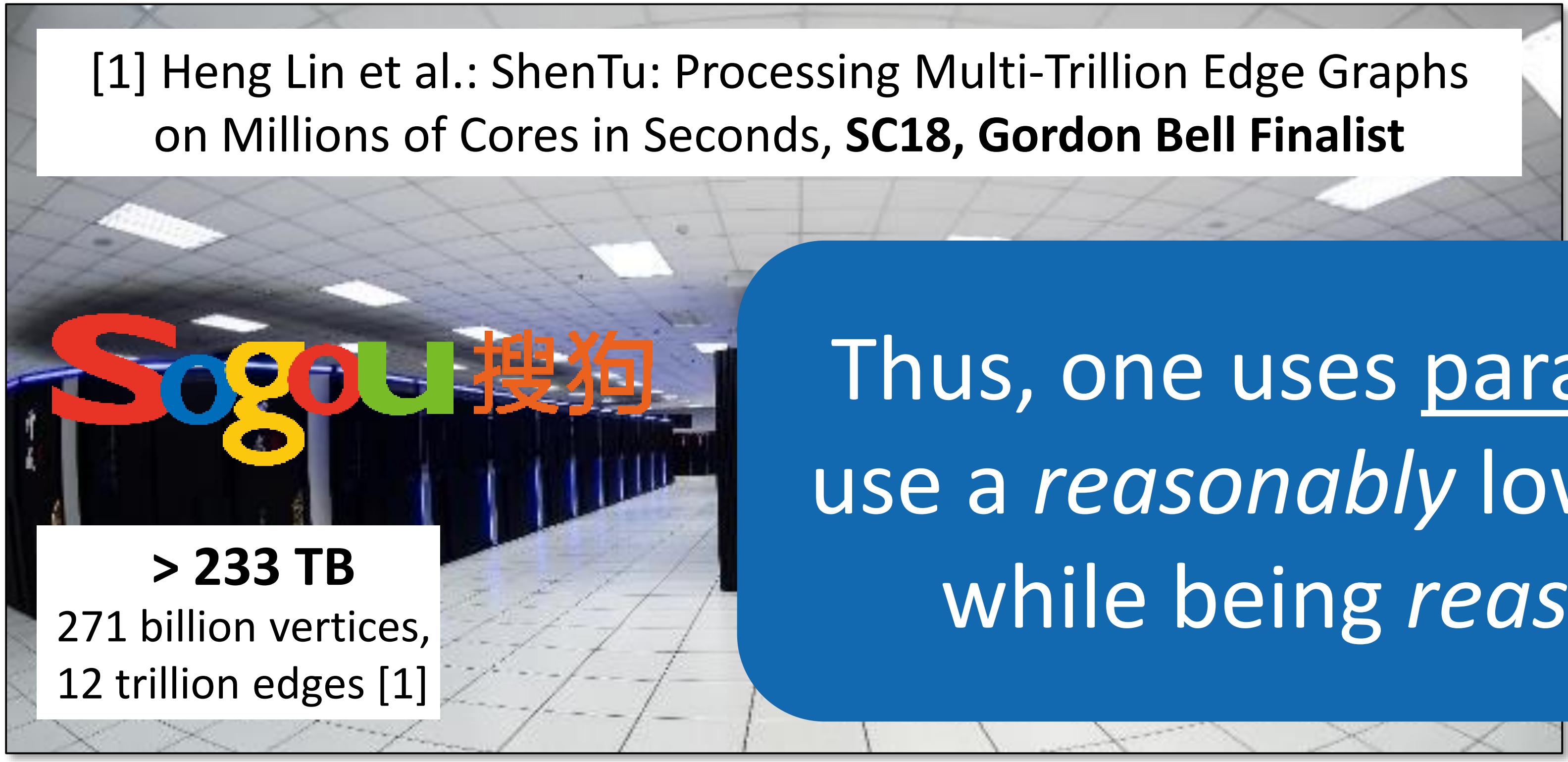

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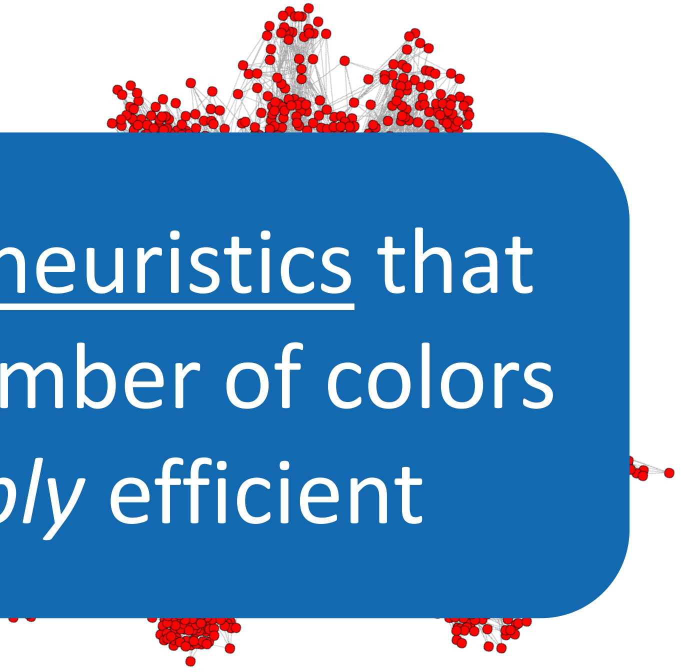

Thus, one uses <u>parallel</u> <u>heuristics</u> that use a *reasonably* low number of colors while being *reasonably* efficient

Parallel graph coloring heuristics

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/* n: number of vertices,
   m: number of edges,
   Δ: maximum vertex degree */
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Goal 1

Maximize quality (i.e., minimize #used colors)

Goal 2

Maximize performance

Both empirically and with theoretical properties (i.e., minimize work, depth, and theoretical limit on #used colors)

Parallel graph coloring heuristics

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Goal 1

Maximize quality (i.e., minimize #used colors)

The order of pi…
vertices impa…
coloring quality

work: total number of operations
depth: longest chain of sequential dependencies

This sounds inherently

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/* n: number of vertices,
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Parallel graph coloring heuristics

A lot of heuristics were introduced, offering different work-depth-quality **tradeoffs**

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Ordering

“First fit” (i.e., any order)

“Largest degree first”

“Smallest degree last”

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“Largest log-degree first”

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Almost all schemes have only trivial quality bounds

Parallel graph coloring heuristics

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A lot of heuristics were introduced, offering different work-depth-quality **tradeoffs**

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Let’s use it as a starting point…

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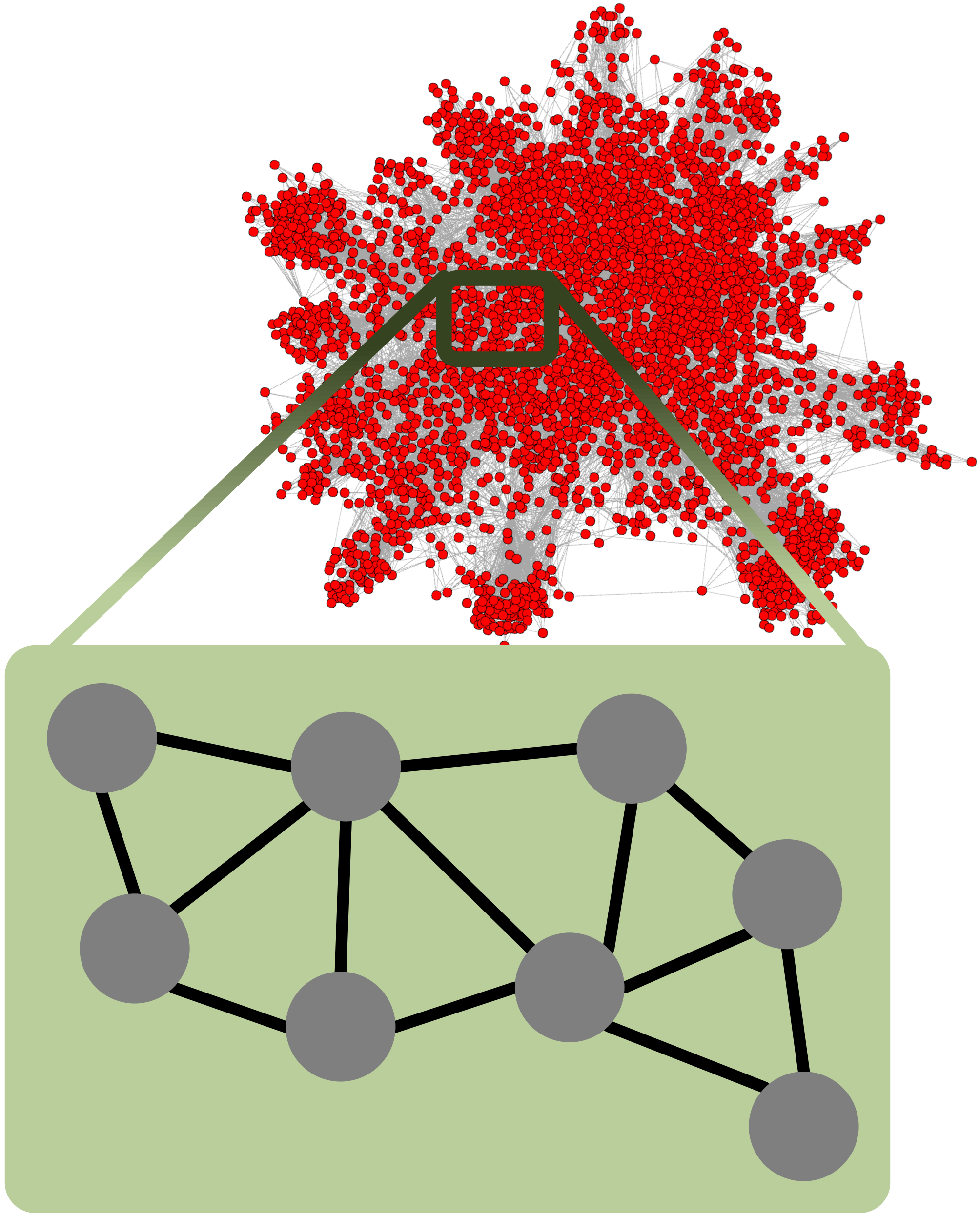

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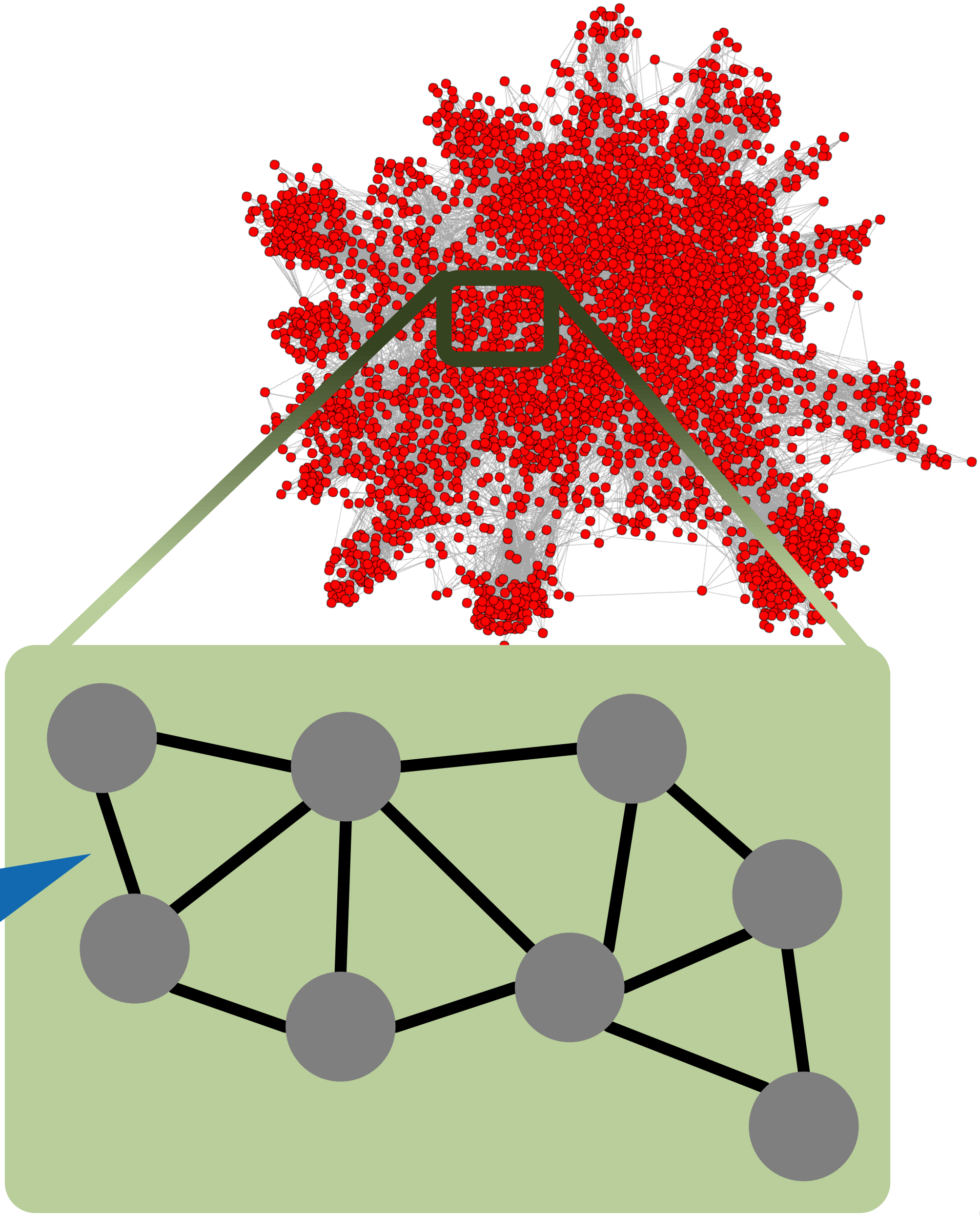

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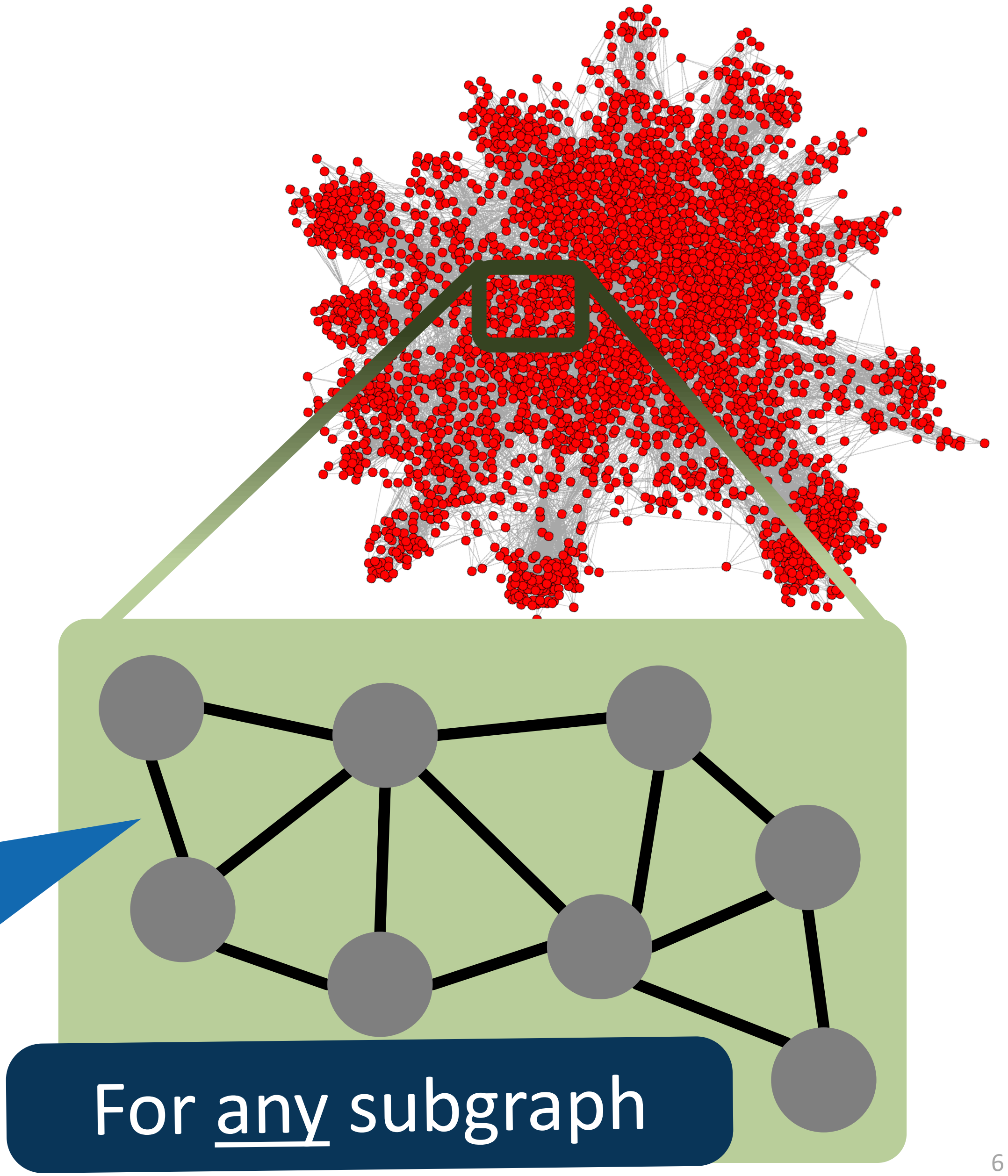

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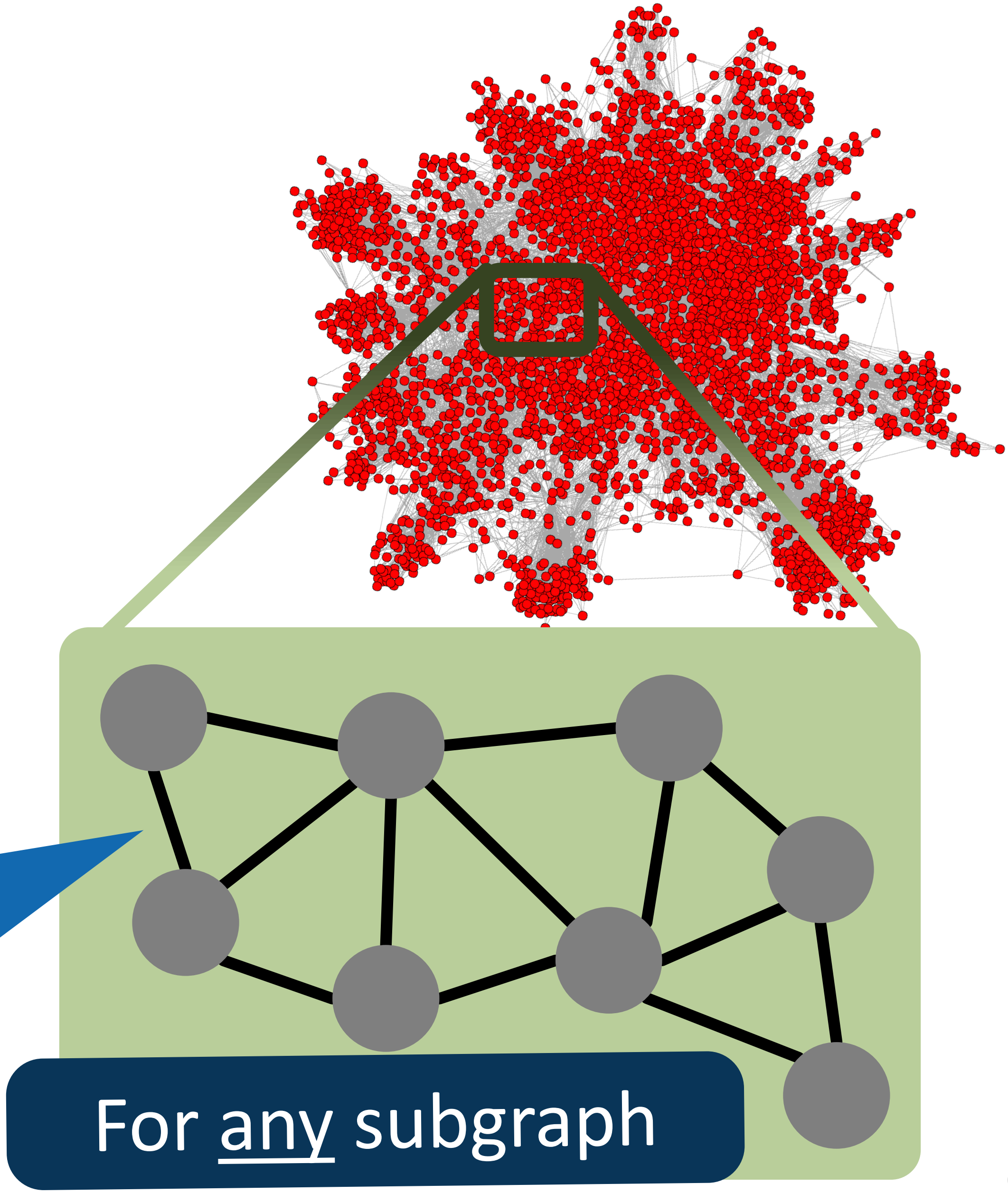

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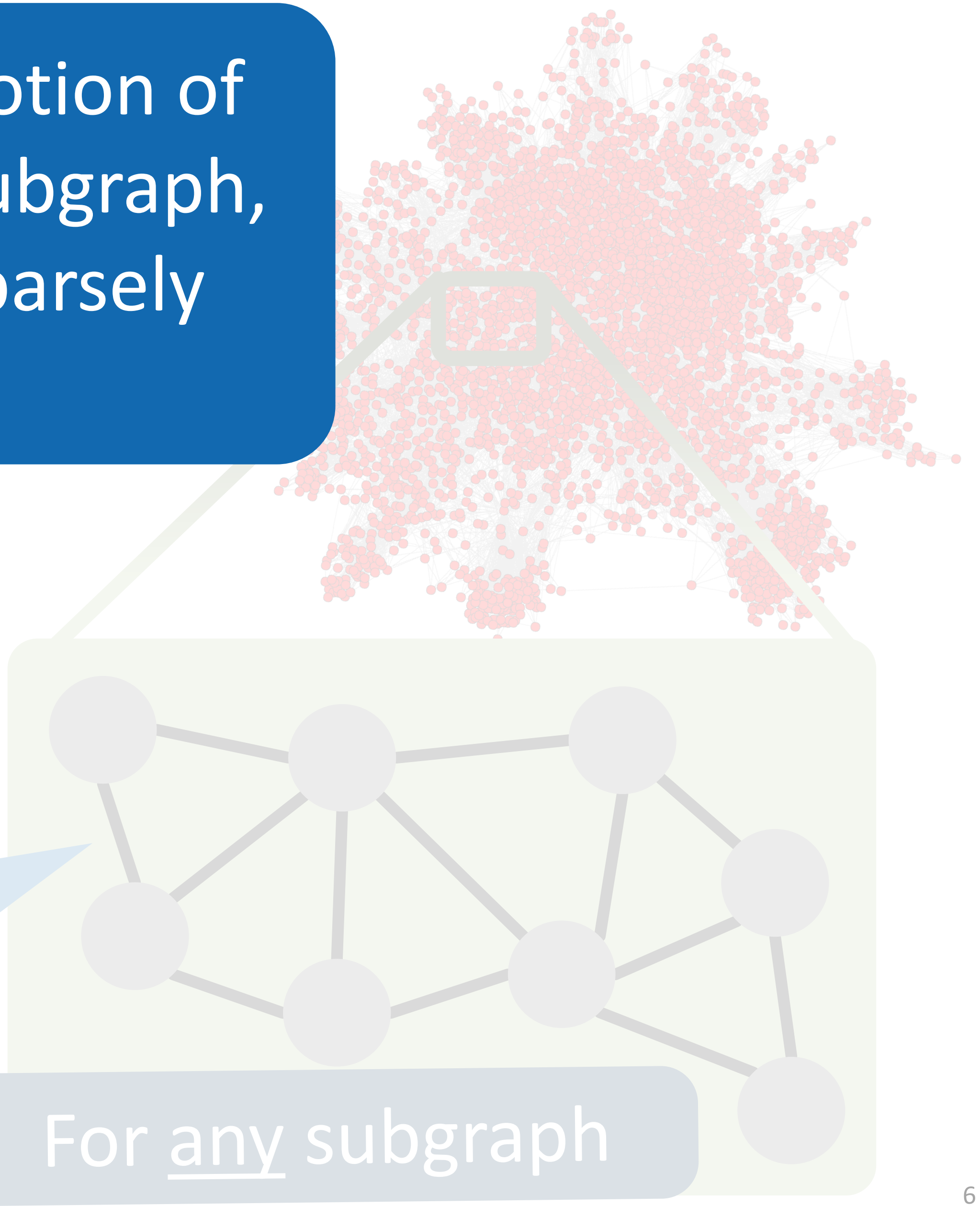

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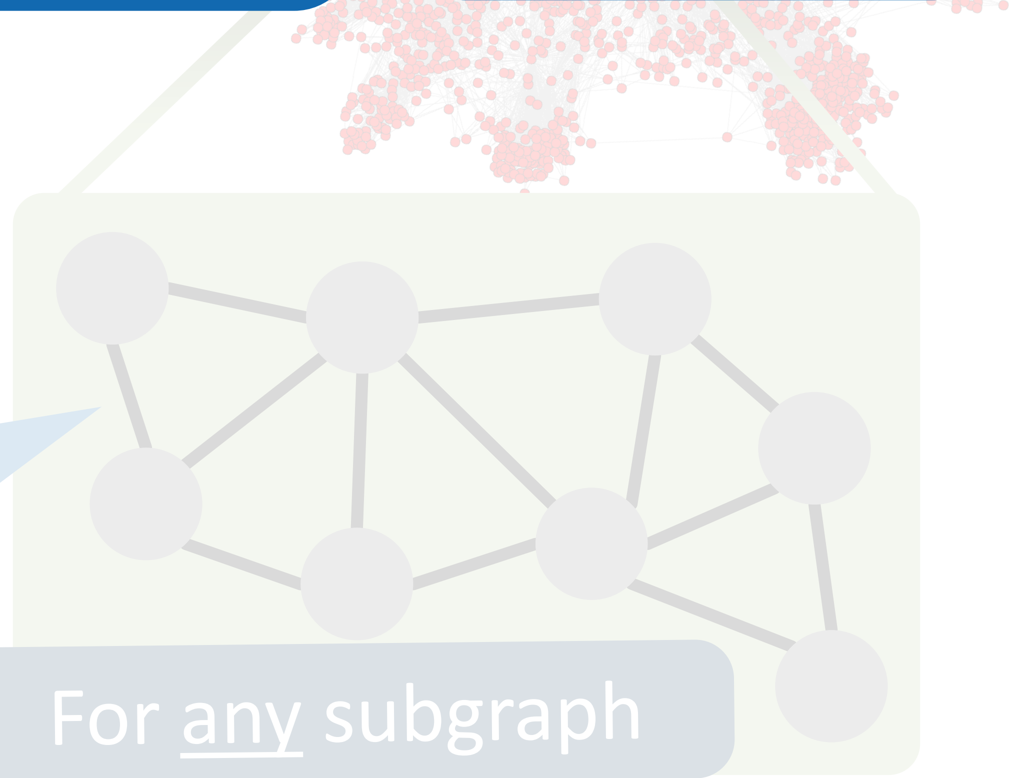

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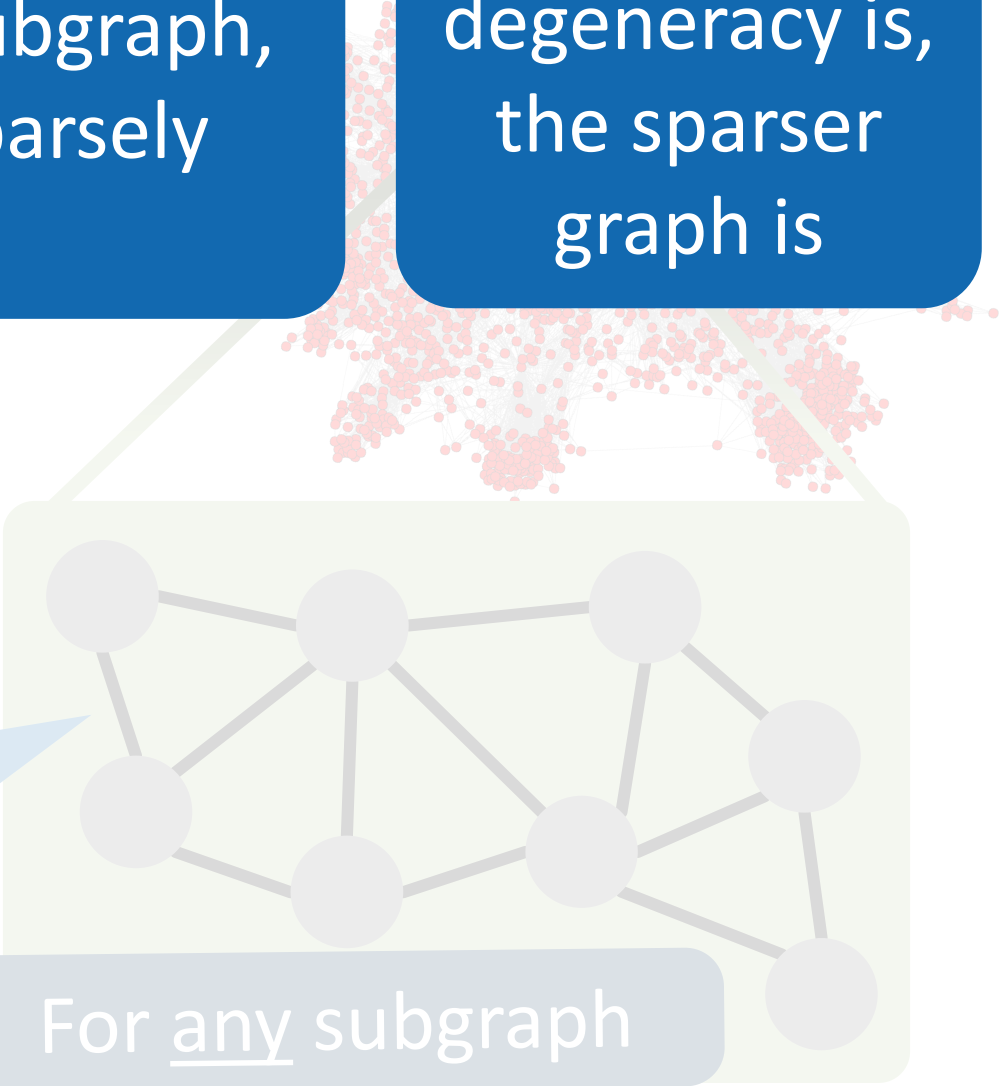

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Great, modern graphs are sparse, so d+1 should be low in practice

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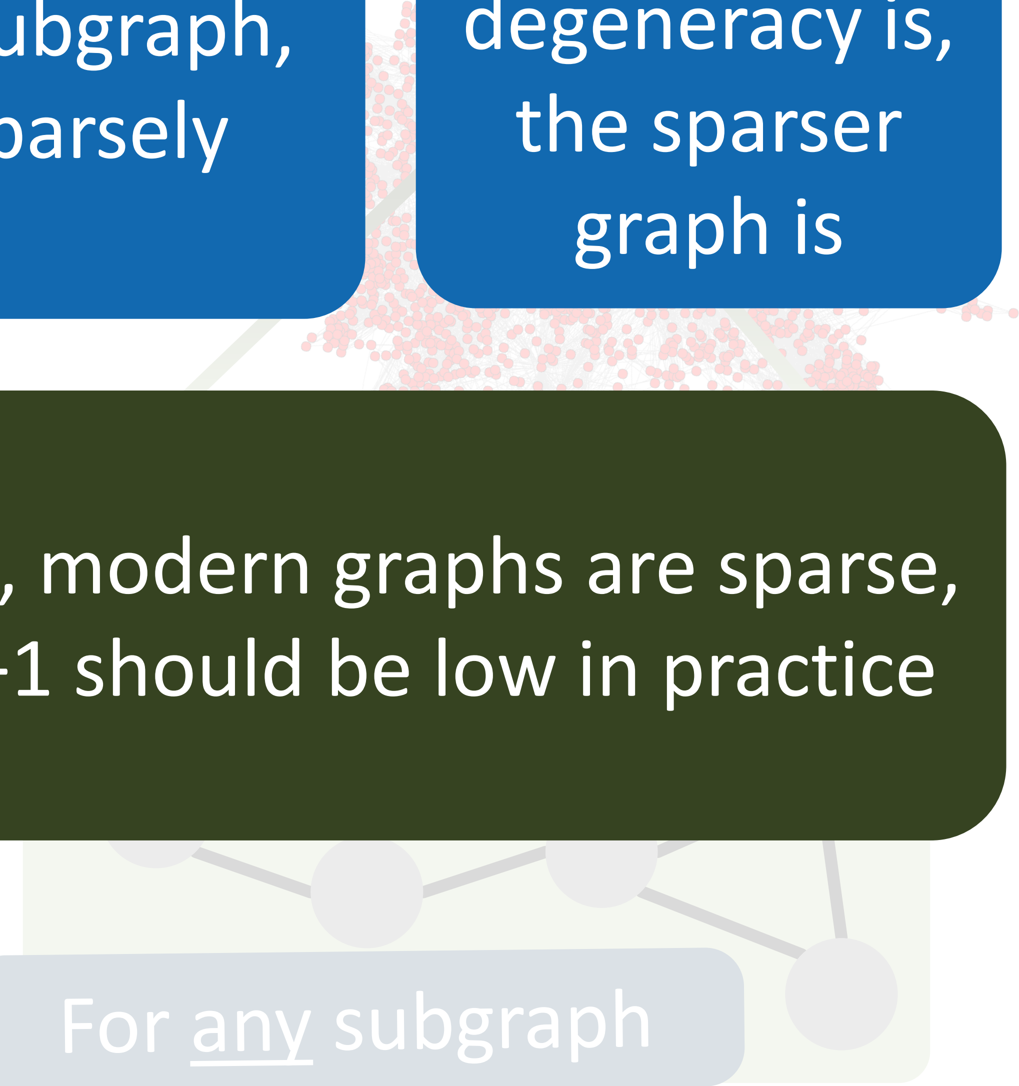

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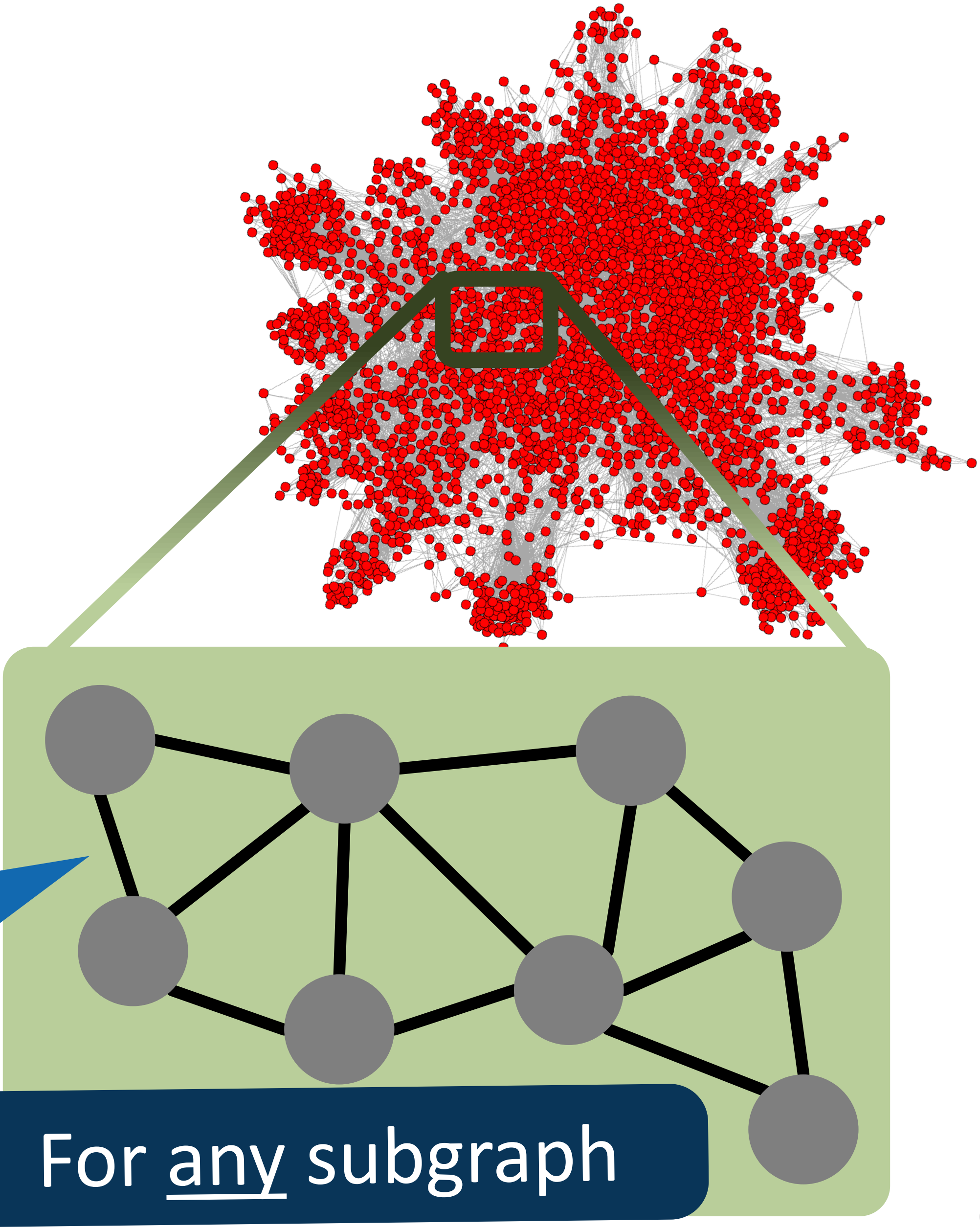

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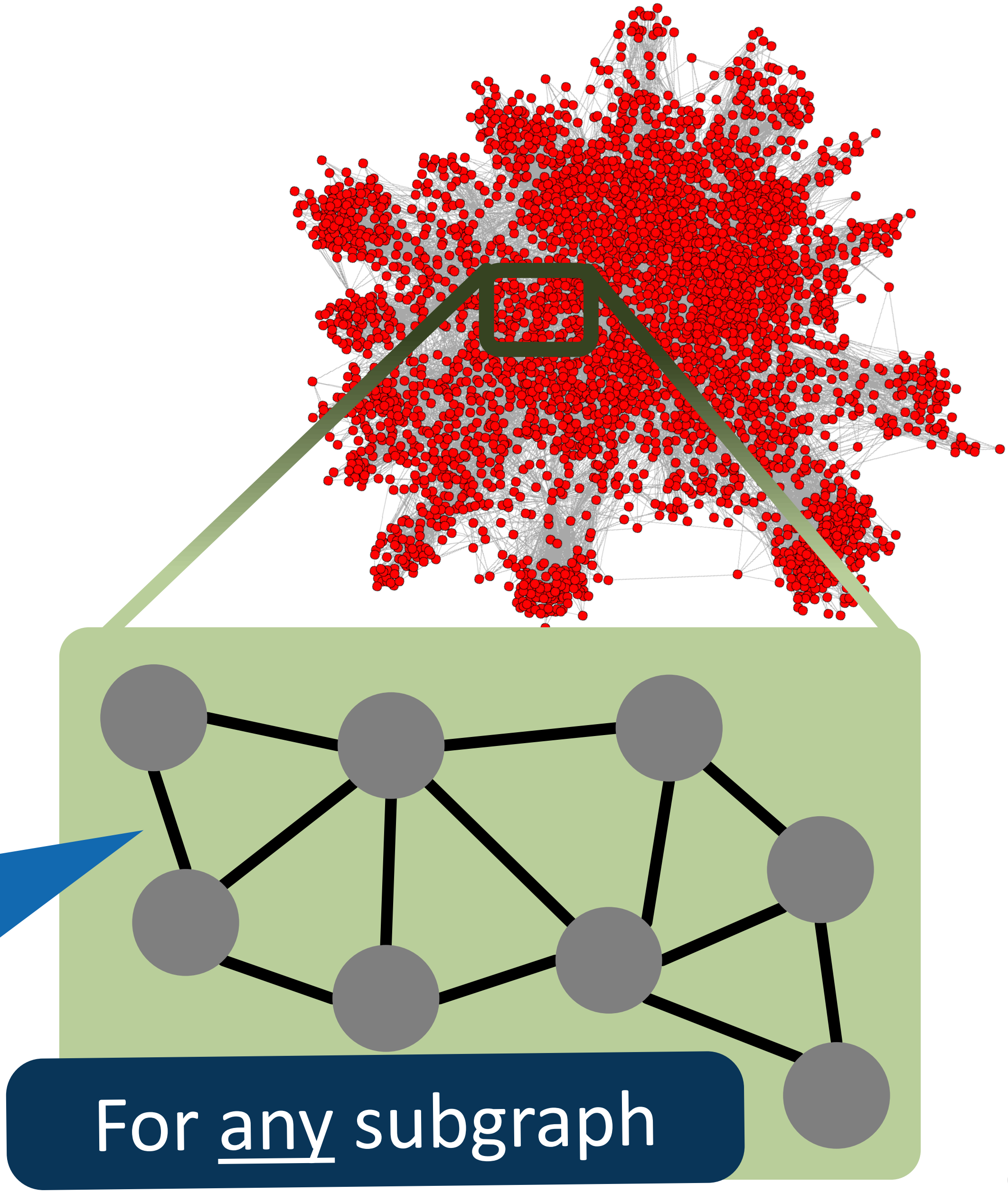

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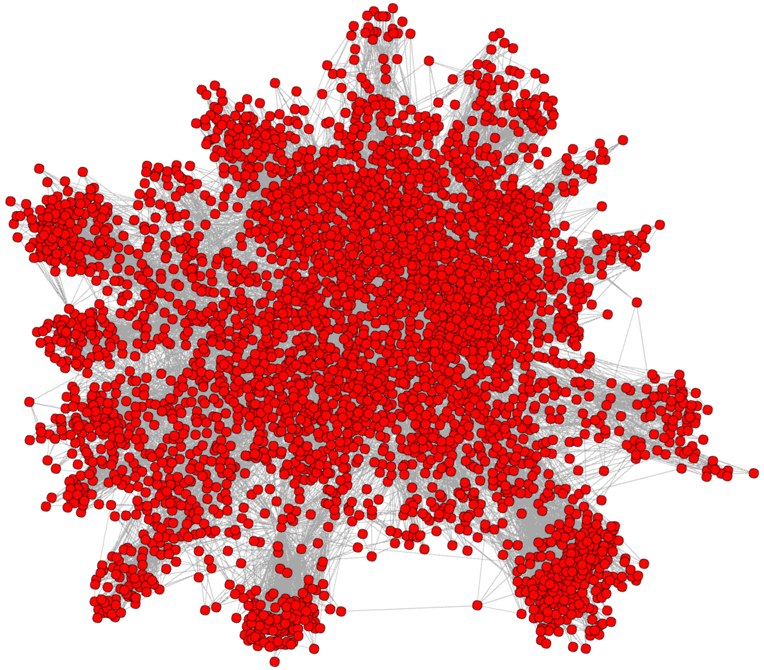

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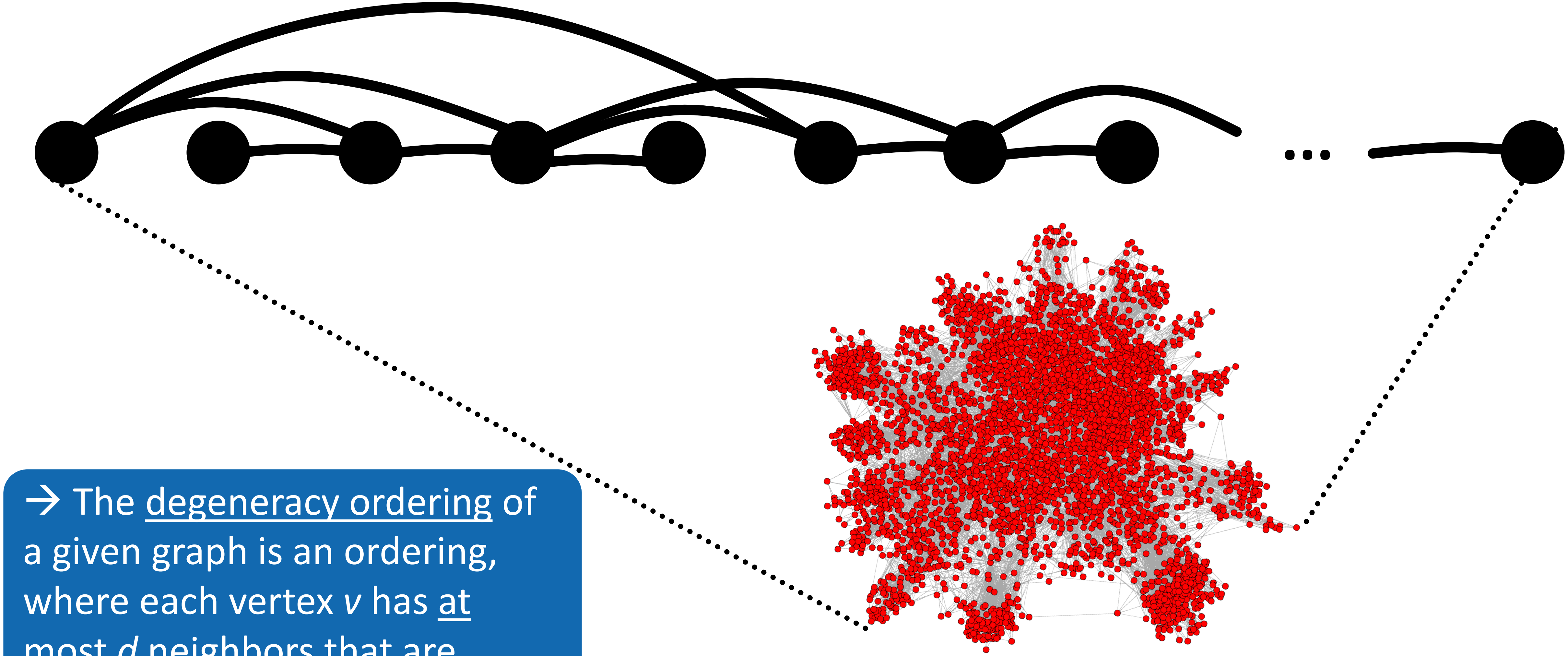

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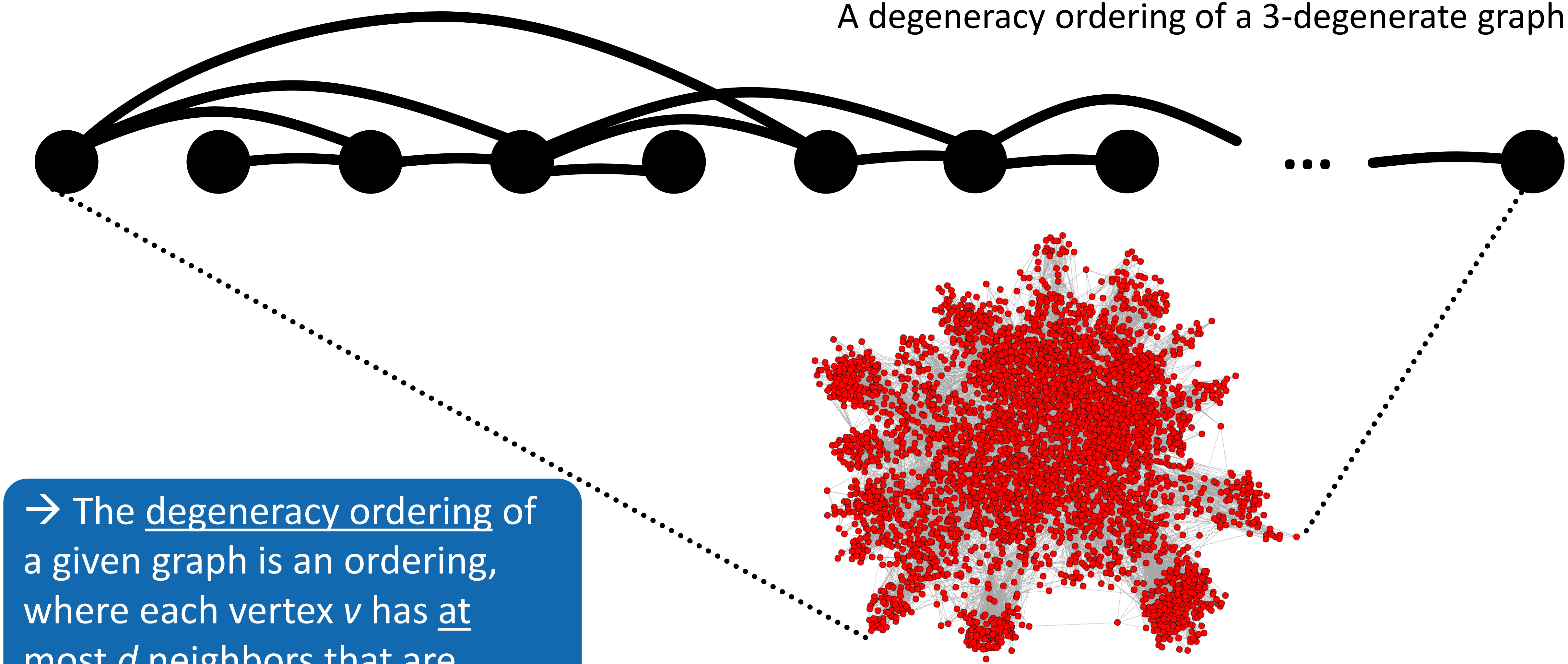

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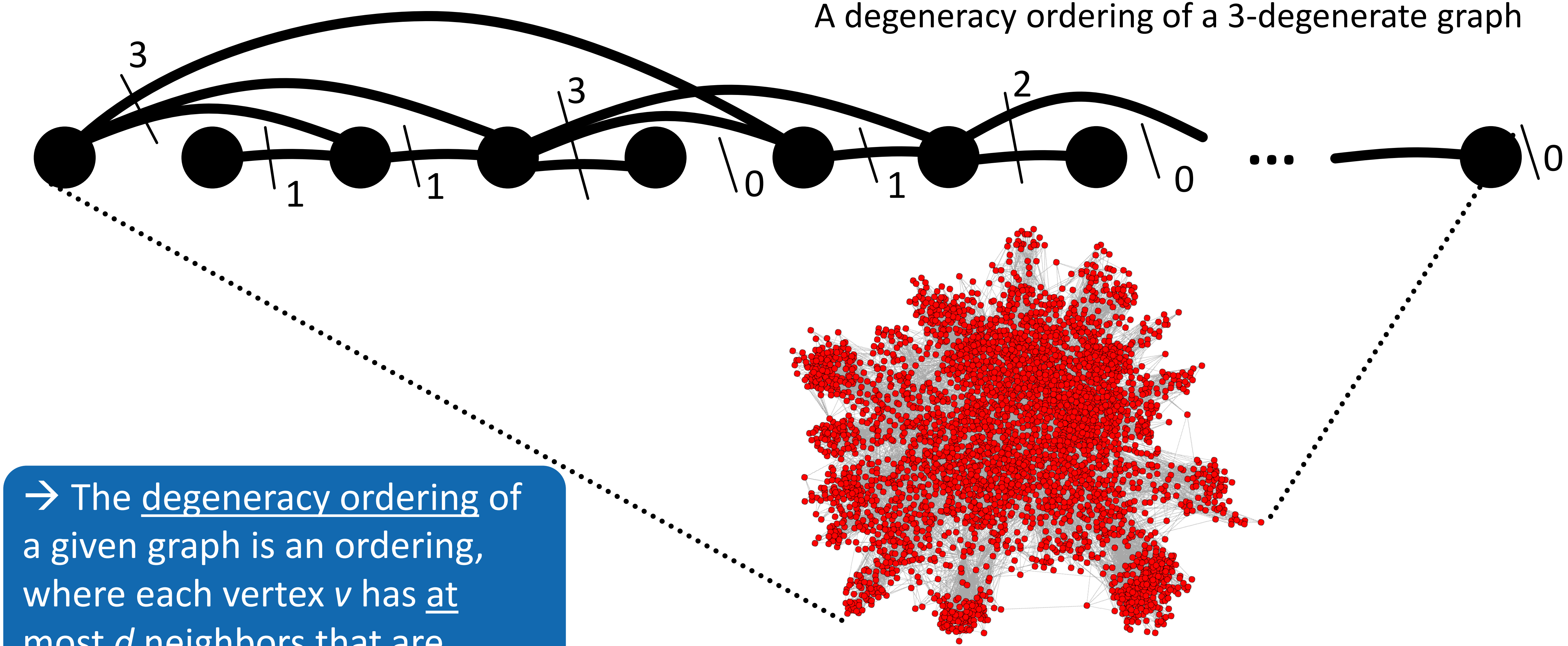
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✗ Deriving the ordering takes O(n) depth (i.e., it is inherently sequential)

✗ The corresponding coloring heuristics is thus bottlenecked by the ordering derivation

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A user-specified parameter that controls a performance-quality tradeoff

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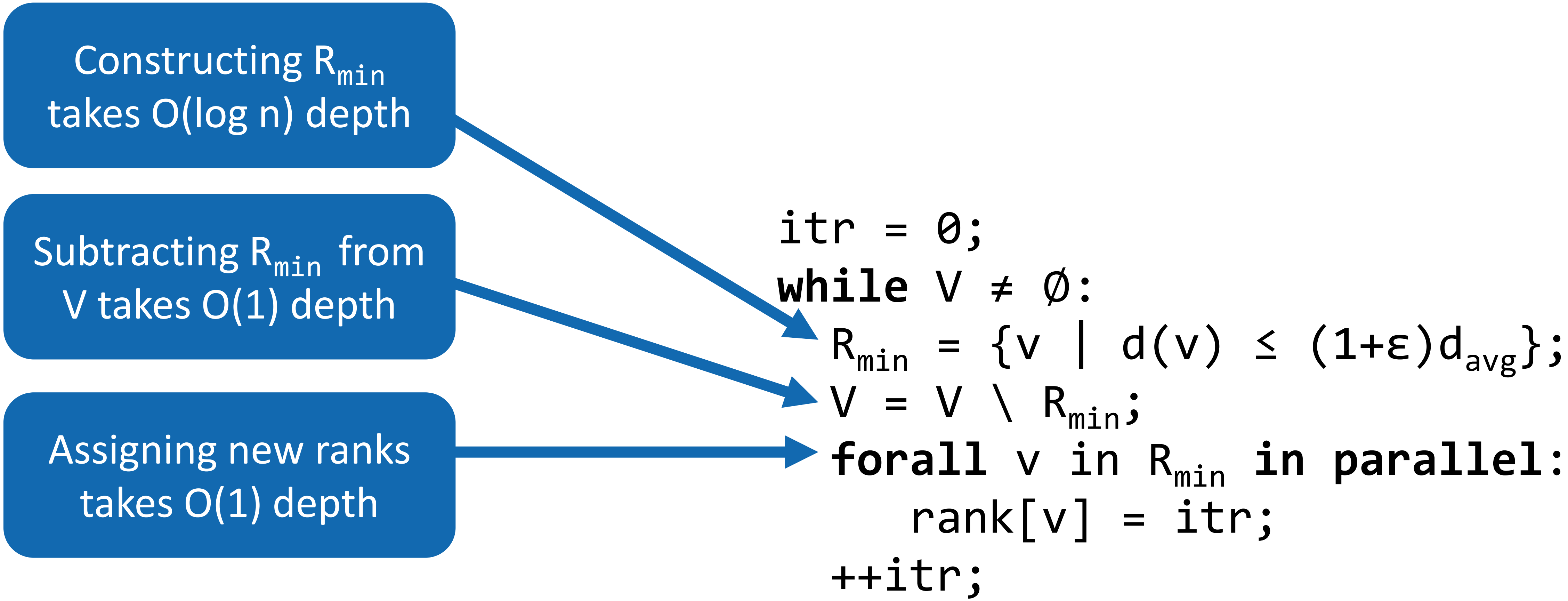

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```
/* n: t              of all vertices
   V: s
   d(v                             ,
   d_avg: average                  /
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All iterations take O(n+m) work

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Approximation: 2(1+ε)

Relaxation approximates the degeneracy by a 2(1+ε) multiplicative factor

Approximate degeneracy ordering

```
/* n: the number of all vertices
   V: set of all vertices,
   d(v): degree of a vertex v,
   d_avg: average degree in V */
```

→ **Key idea**: try a relaxation of the strict degeneracy order, at the cost of (some) accuracy loss.

Strict degeneracy order:

ADG: approximate deg

Work: O(

Depth: O

How does it impact the actual graph coloring?

2(1+ε)

Relaxation approximates the degeneracy by a 2(1+ε) multiplicative factor

Parallel graph coloring heuristics

Parallel graph coloring heuristics

Let's see how the coloring heuristic uses the orderings

Parallel graph coloring heuristics

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Parallel graph coloring heuristics

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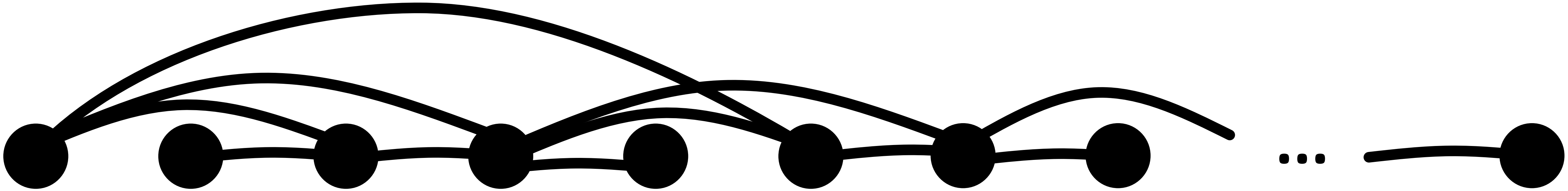
Let’s see how the coloring heuristic uses the orderings

→ The degeneracy ordering of a given graph is an ordering, where each vertex v has at most d neighbors that are ordered higher than v (d is G’s degeneracy).



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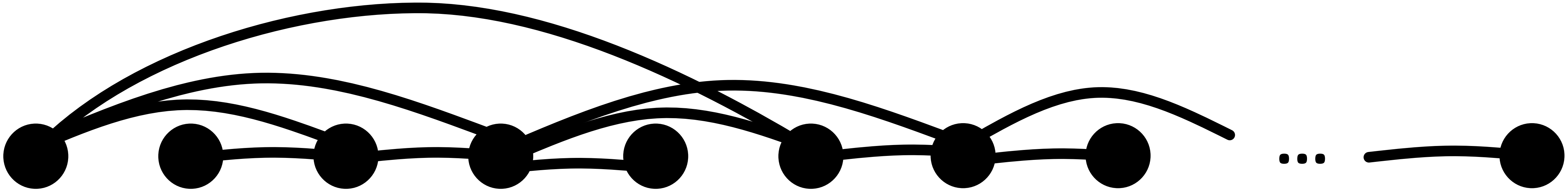

A degeneracy ordering of a 3-degenerate graph

Parallel graph coloring heuristics

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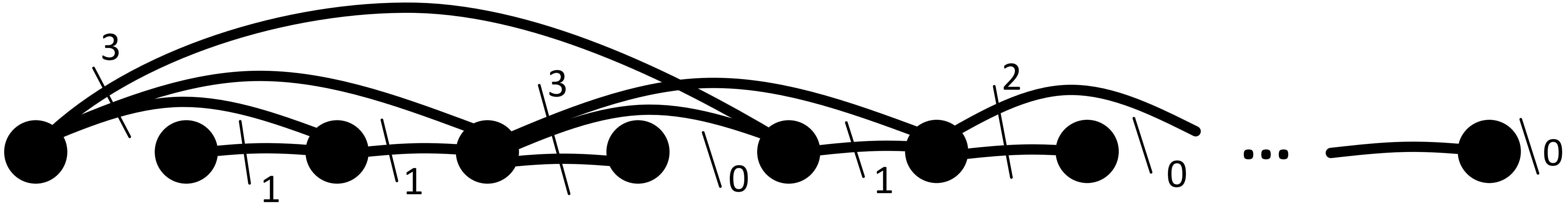

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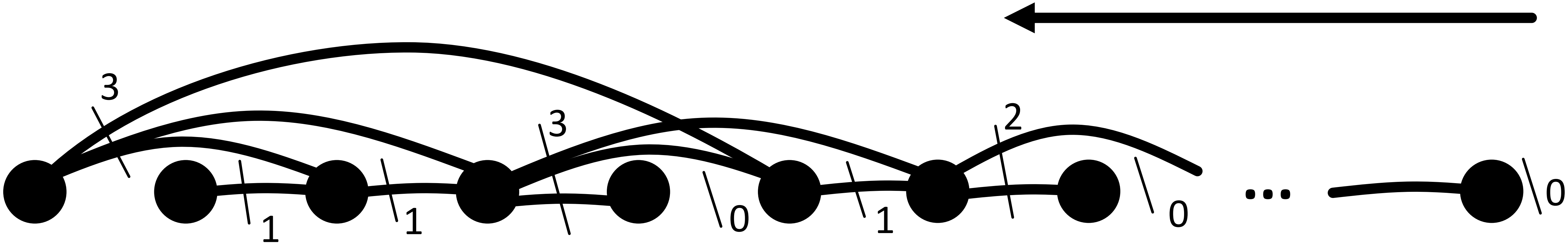

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Let's see how the coloring heuristic uses the orderings

→ The degeneracy ordering of a given graph is an ordering, where each vertex v has at most d neighbors that are ordered higher than v (d is G's degeneracy).

Using the strict degeneracy ordering, we get at most d+1 colors

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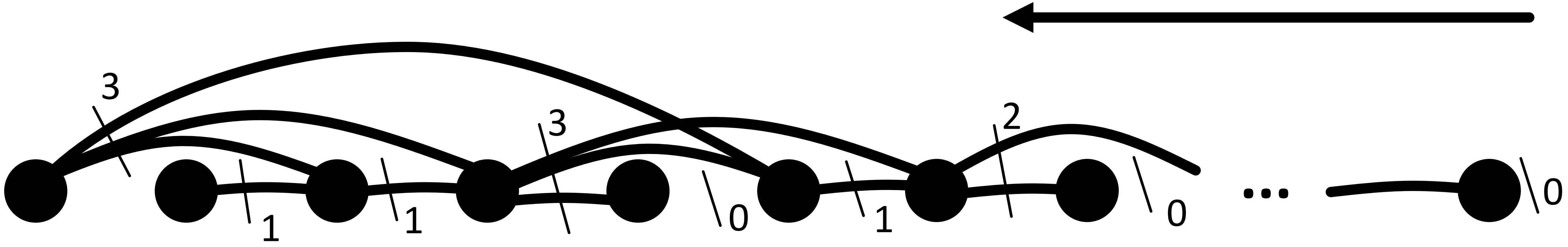

A degeneracy ordering of a 3-degenerate graph

Parallel graph coloring heuristics + ADG

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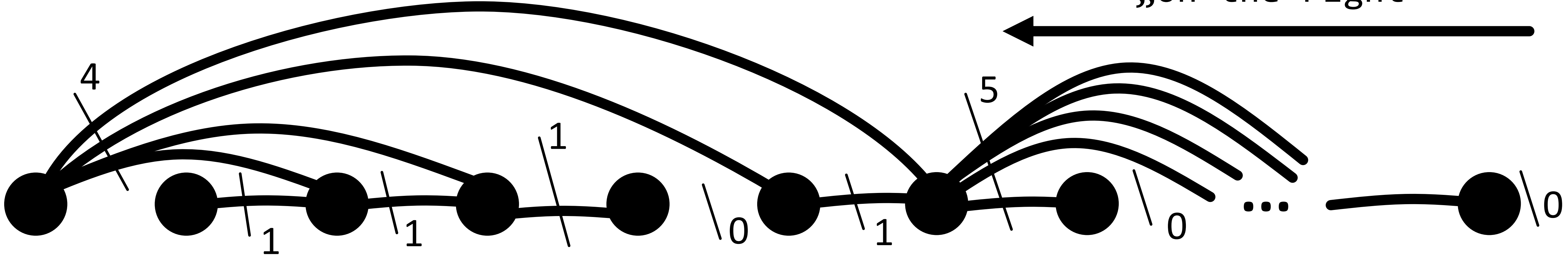

A 2(1+ε)-approximate degeneracy ordering of a 3-degenerate graph

Parallel graph coloring heuristics + ADG

In the ADG ordering, each vertex v has at most 2(1+ε)d neighbors that are ordered higher than v

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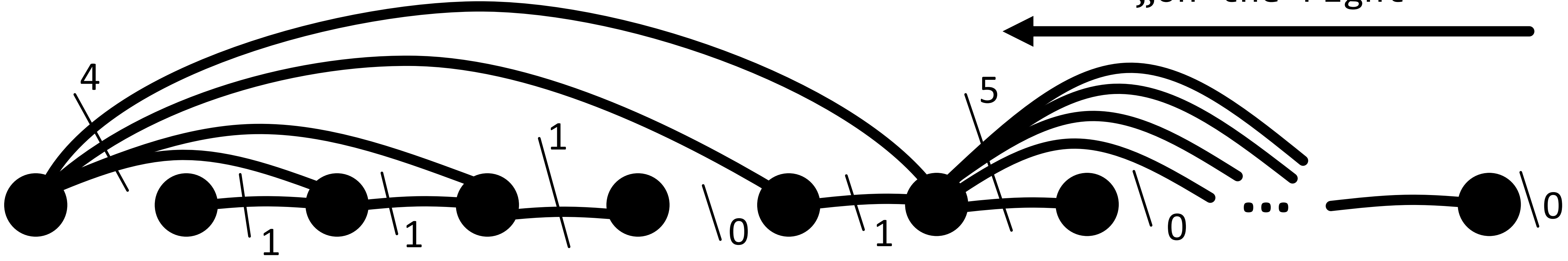

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Parallel graph coloring heuristics + ADG

In the ADG ordering, each vertex v has at most 2(1+ε)d neighbors that are ordered higher than v

Using ADG, we get at most 2(1+ε)d + 1 colors

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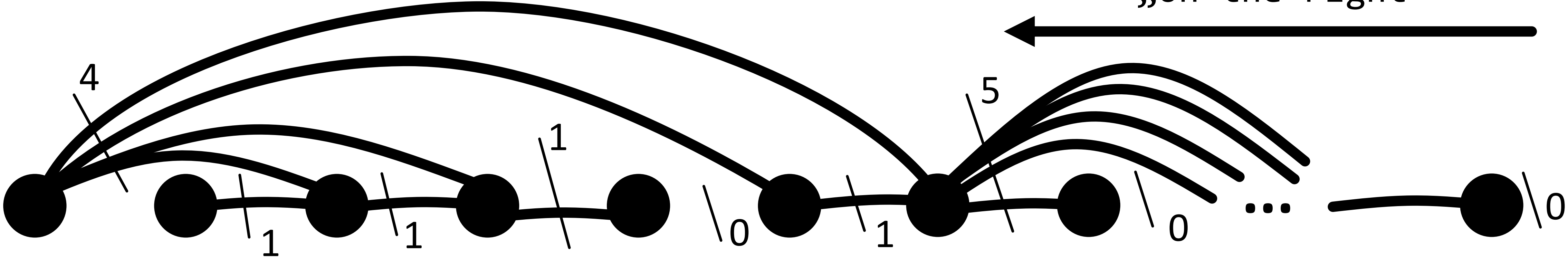

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Parallel graph coloring heuristics + ADG

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Parallel graph coloring heuristics + ADG

In the ADG ordering, each vertex v has at most 2(1+ε)d neighbors that are ordered higher than v

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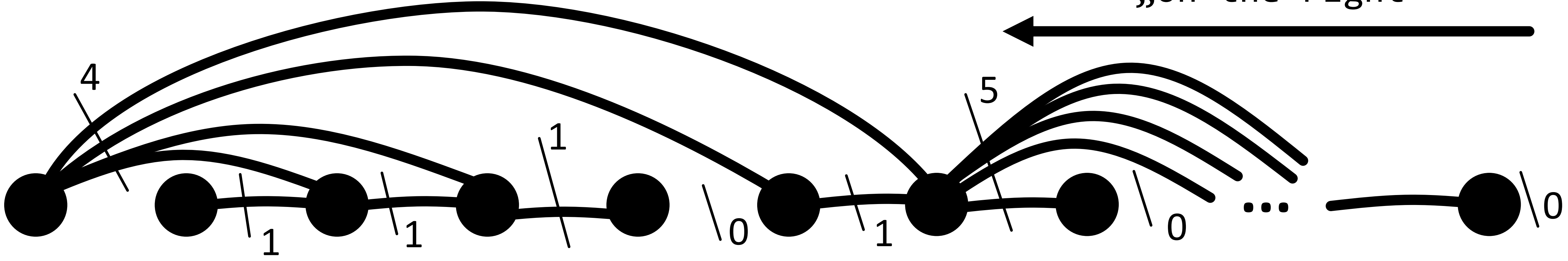

A 2(1+ε)-approximate degeneracy ordering of a 3-degenerate graph

Parallel graph coloring heuristics + ADG

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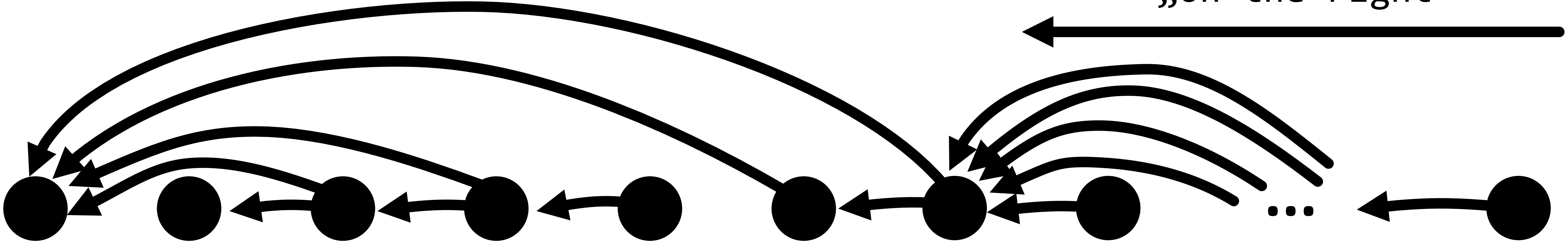
We consider a DAG imposed over the input graph G, with directions assigned based on the used vertex ordering



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Now, it was proved that a parallel coloring heuristics runs in O(|P| log Δ + log n) depth and O(n+m) work [1].

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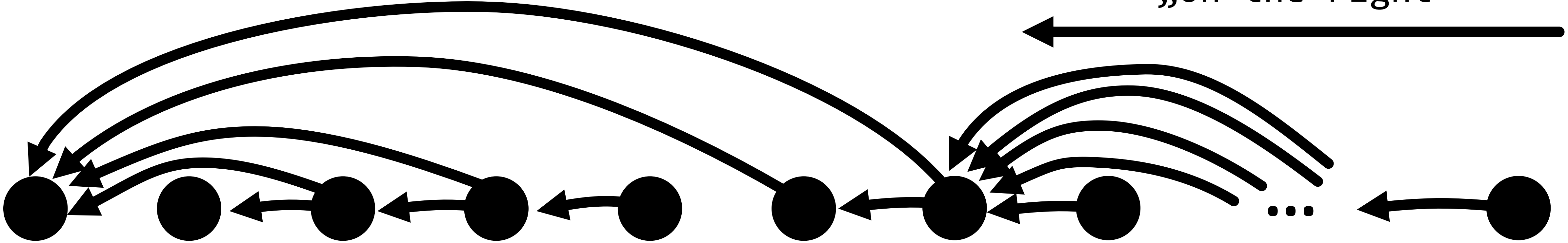

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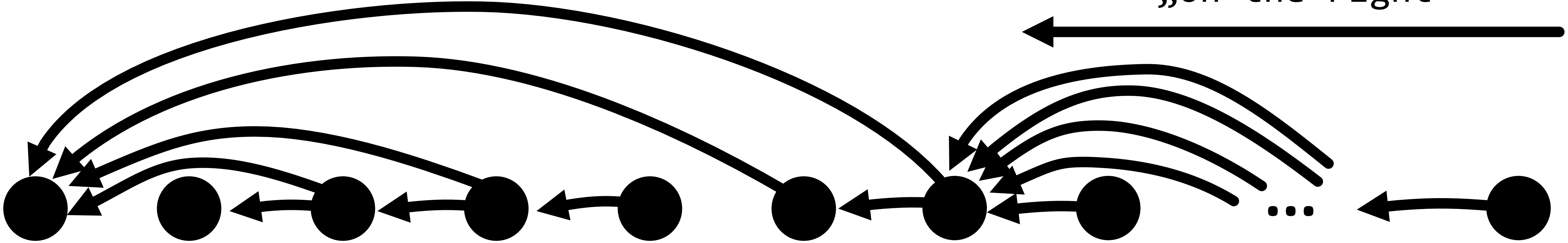

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Let’s see some intuition

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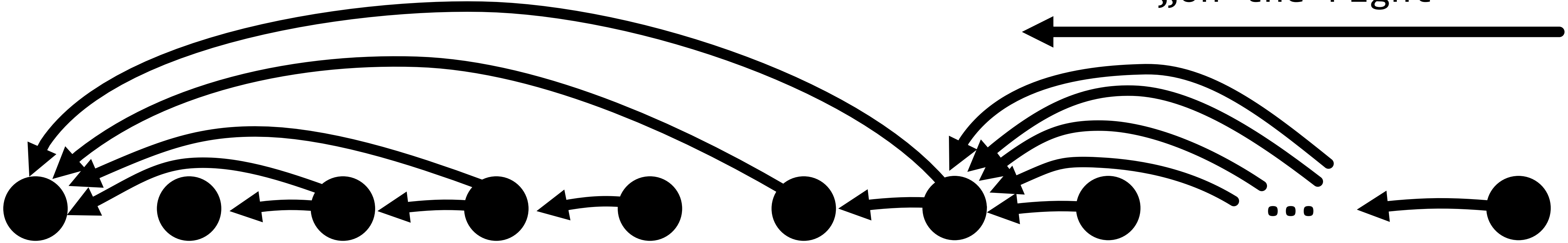

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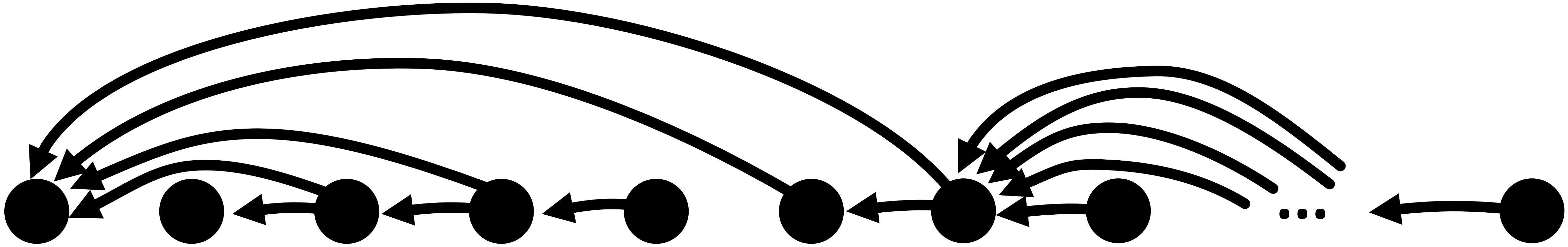
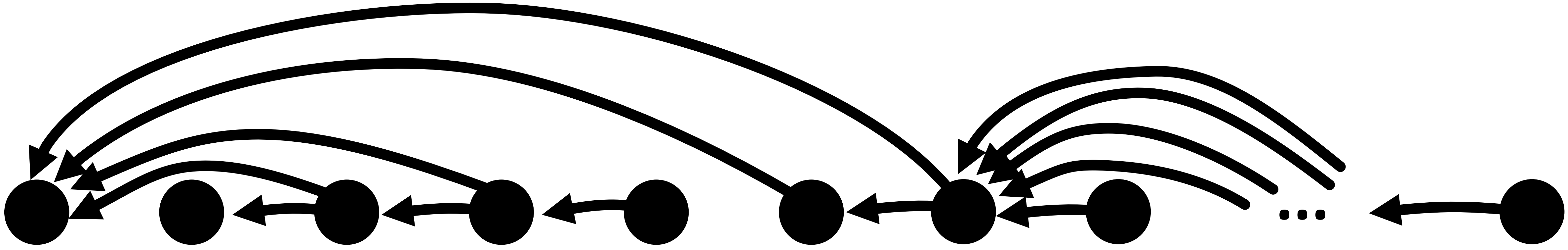


A 2(1+ε)-approximate degeneracy ordering of a 3-degenerate graph

Parallel graph coloring heuristics + ADG

Vertices with the same ADG rank form subgraphs



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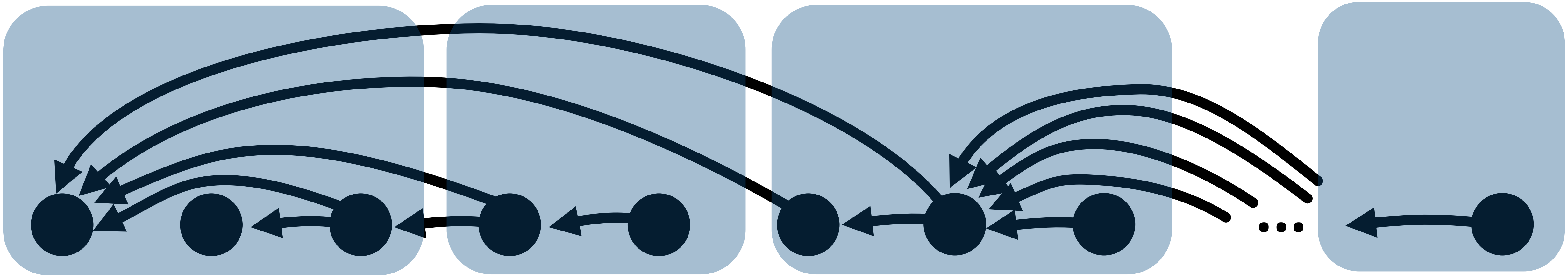

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Parallel graph coloring heuristics + ADG

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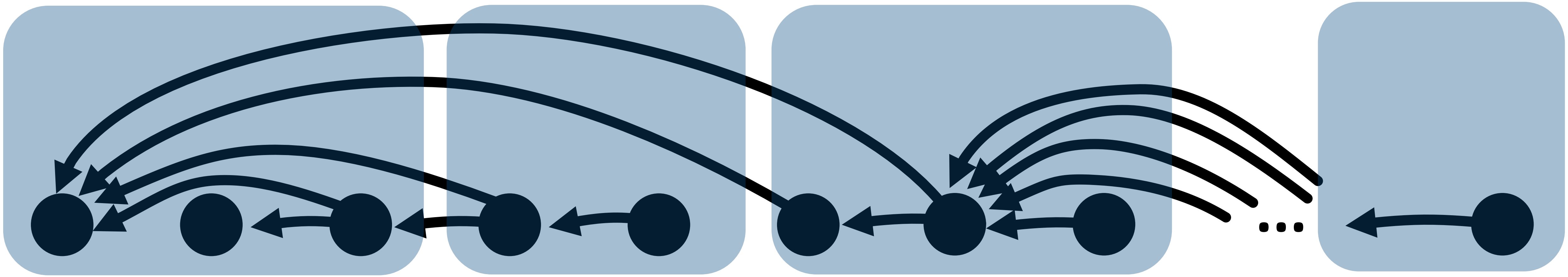
Analyze |P| by analyzing the lengths of its parts, going via each subgraph



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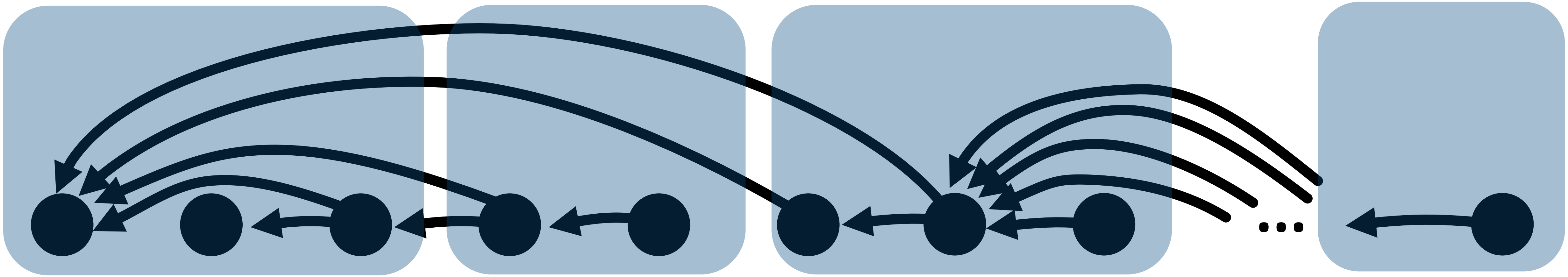

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Parallel graph coloring heuristics + ADG

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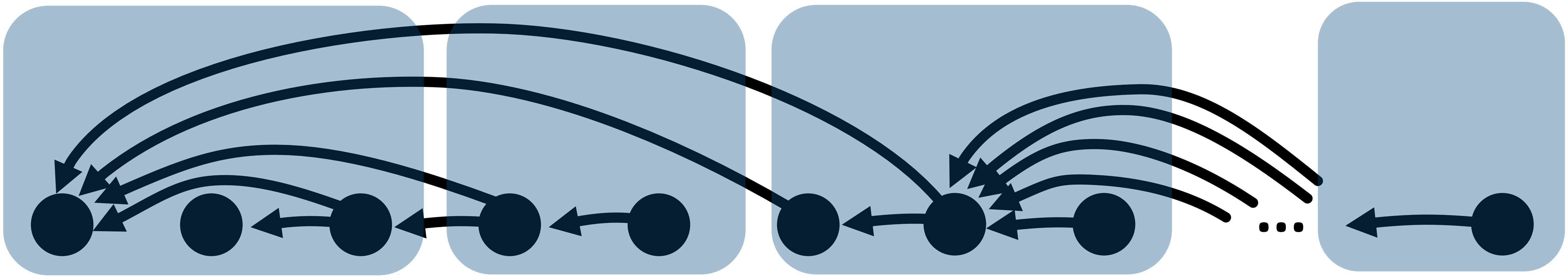
“There is only as far (constant) as you can go in a subgraph”



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ADG	$\mathbb{E}\, O\left(\log^2 n + \log\Delta \cdot \left(d\log n + \frac{\log d \cdot \log^2 n}{\log\log n}\right)\right)$	$O(n+m)$	$2(1+\varepsilon)d+1$

Parallel graph coloring heuristics

```
/* n: number of vertices,
   m: number of edges,
   Δ: maximum vertex degree,
   d: graph's degeneracy */
```

Ordering	Depth	Work	Quality
"First fit" (i.e., any order)	No general bounds; $\Omega(n)$ for some graphs	$O(n+m)$	$\Delta+1$
"Largest degree first"	No general bounds; $\Omega(\Delta^2)$ for some graphs	$O(n+m)$	$\Delta+1$
"Smallest degree last"	No general bounds; $\Omega(n)$ for some graphs	$O(n+m)$	$d+1$
Random	$\mathbb{E}\, O\left(\frac{\log n}{\log\log n}\right)$	$O(n+m)$	$\Delta+1$
Random	$\mathbb{E}\, O\left(\log n + \log\Delta \cdot \min\left\{\sqrt{m}, \Delta + \frac{\log\Delta \log n}{\log\log n}\right\}\right)$	$O(n+m)$	$\Delta+1$
"Largest log-degree first"	$\mathbb{E}\, O\left(\log n + \log\Delta \cdot \left(\min\{\Delta, \sqrt{m}\} + \frac{\log^2\Delta \log n}{\log\log n}\right)\right)$	$O(n+m)$	$\Delta+1$
"Smallest log-degree last"	$\mathbb{E}\, O\left(\log\Delta \log n + \log\Delta \cdot \left(\min\{\Delta, \sqrt{m}\} + \frac{\log^2\Delta \log n}{\log\log n}\right)\right)$	$O(n+m)$	$\Delta+1$
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Parallel graph coloring heuristics

```
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   m: number of edges,
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```

Anything else?

Ordering	**Depth**	**Work**	**Quality**
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“Largest degree first”	No general bounds; $\Omega(\Delta^2)$ for some graphs	$O(n+m)$	$\Delta+1$
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ADG ✓	$\mathbb{E}\, O\left(\log^2 n + \log\Delta \cdot \left(d \log n + \frac{\log d \cdot \log^2 n}{\log\log n}\right)\right)$	✓ $O(n+m)$	$2(1+\varepsilon)d+1$

Parallel graph coloring heuristics

Based on „speculative coloring”

Parallel graph coloring heuristics

Based on „speculative coloring”

→ Construct the ADG-induced partitioning

Parallel graph coloring heuristics

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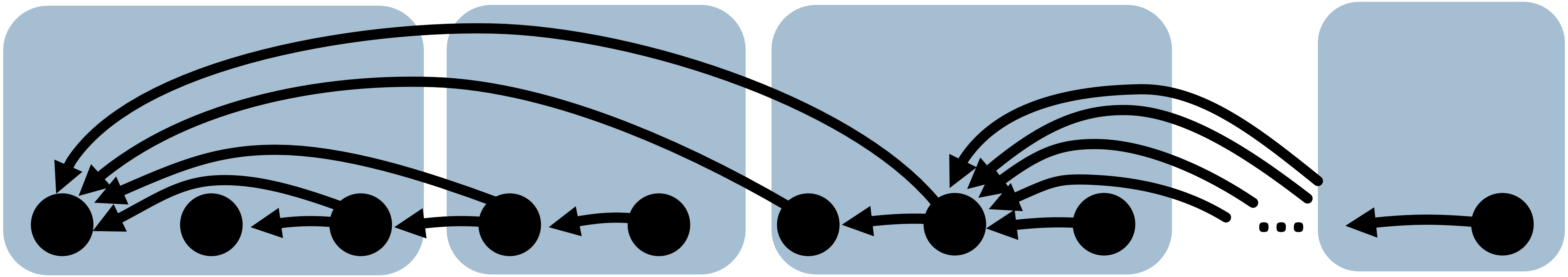
→ Construct the ADG-induced partitioning

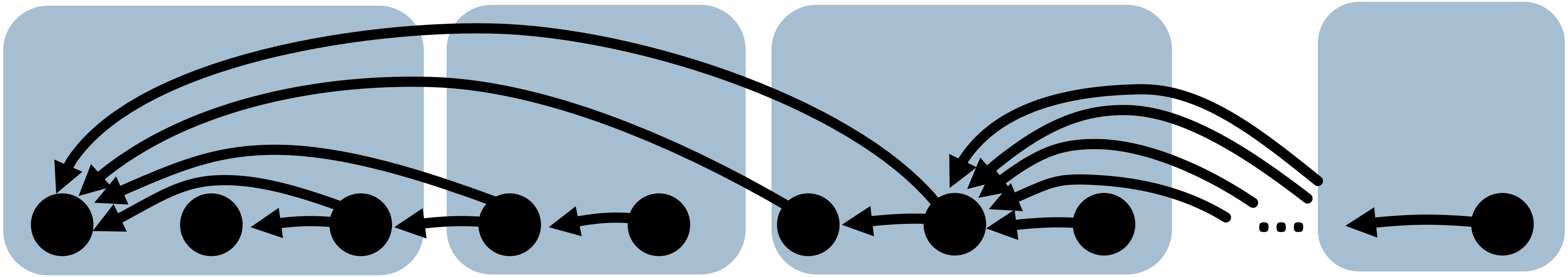


Parallel graph coloring heuristics

Based on „speculative coloring”

→ Construct the ADG-induced partitioning

→ Now, color each partition independently (“speculative coloring”)



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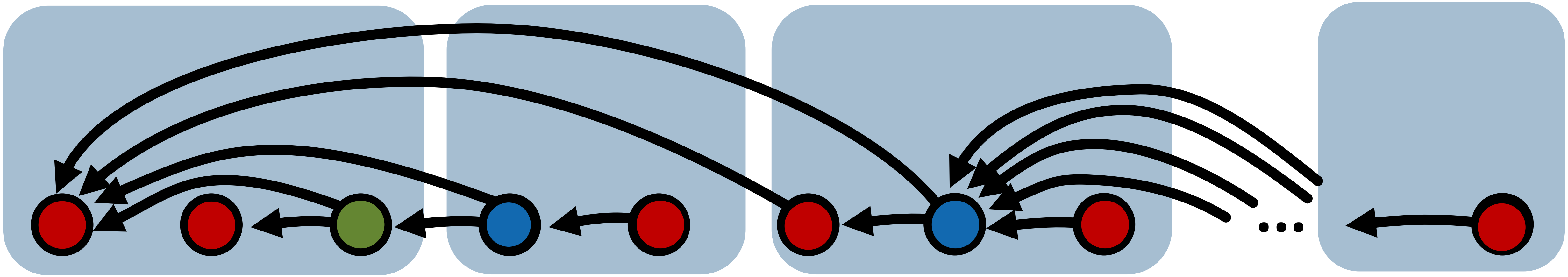

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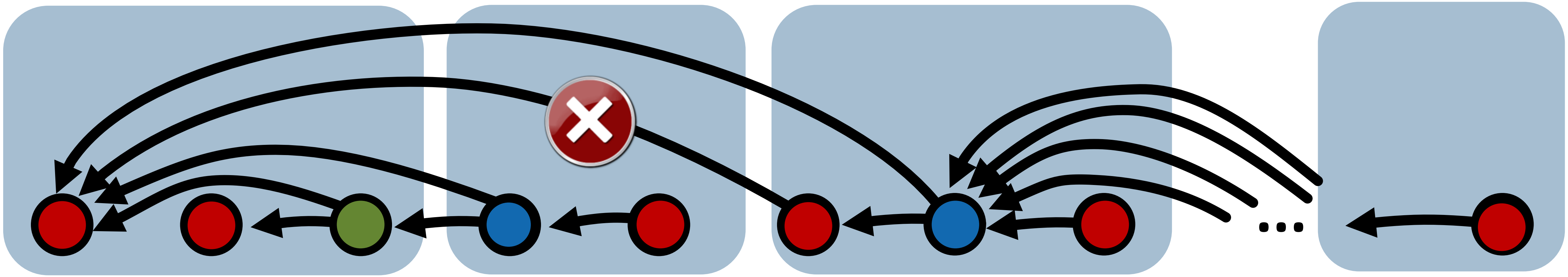


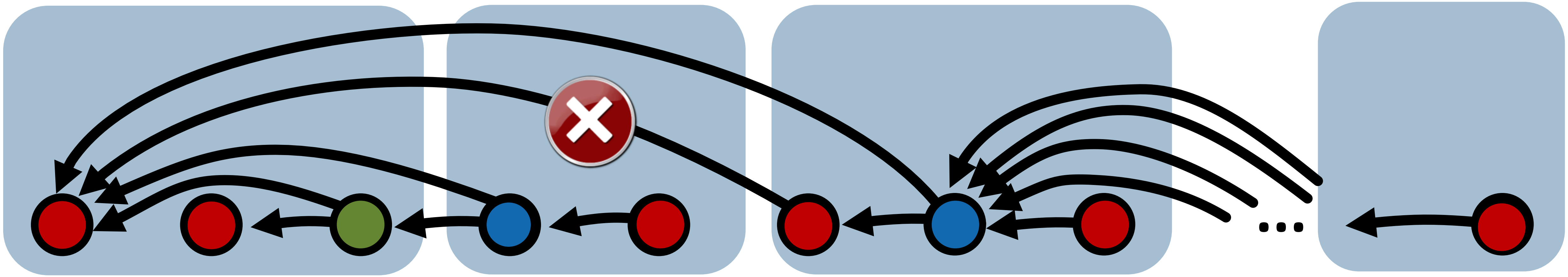
Parallel graph coloring heuristics

Based on „speculative coloring”

→ Construct the ADG-induced partitioning

→ Now, color each partition independently (“speculative coloring”)

→ Any coloring „conflicts” (vertices with the same colors) are by repeating the coloring on conflicting vertices as many times as needed



Parallel graph coloring heuristics

Based on „speculative coloring”

→ Construct the ADG-induced partitioning

→ Now, color each partition independently (“speculative coloring”)

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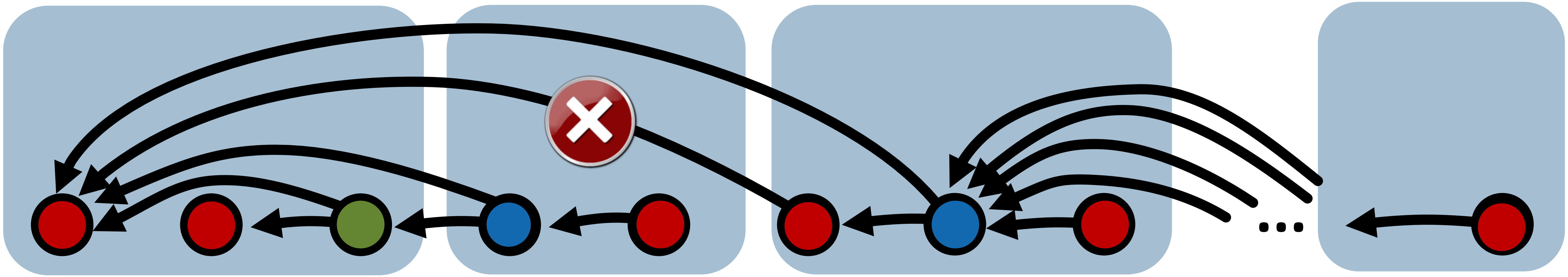
→ Each such partition is “low-degree”: it has a bounded number of edges to any other such partitions (by the definition of ADG)



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ADG (scheduling)	$\mathbb{E}\ O\left(\log^2 n+\log\Delta\cdot\left(d\log n+\frac{\log d\cdot\log^2 n}{\log\log n}\right)\right)$	$O(n+m)$	$2(1+\varepsilon)d+1$

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Parallel graph coloring heuristics

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ADG (speculative)

All details, proofs, etc., are in the paper ☺

Evaluation

Evaluation

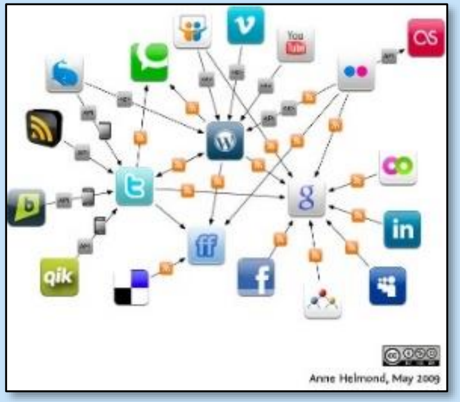
Graphs

Evaluation

Graphs

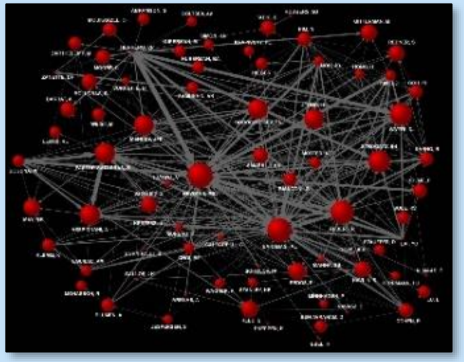
Road networks

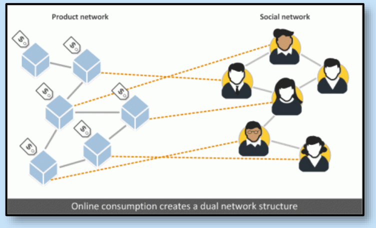
Social networks



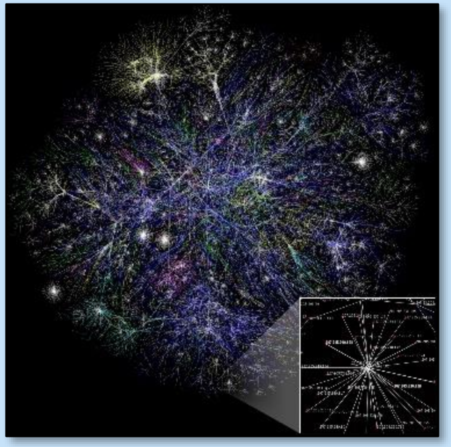
Communication graphs

Citation & collaboration graphs





Purchase networks



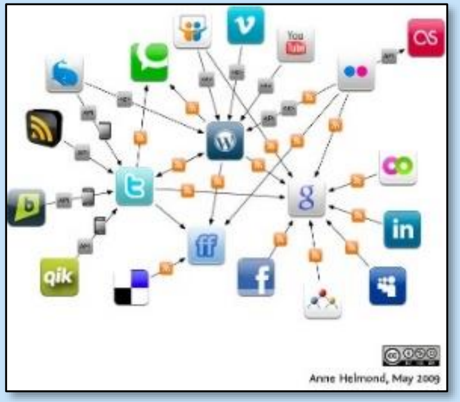
Web graphs

Evaluation

Graphs

Road networks

Social networks



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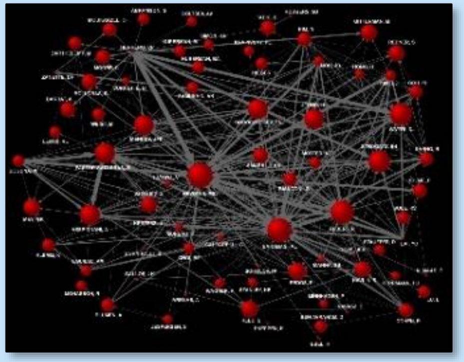

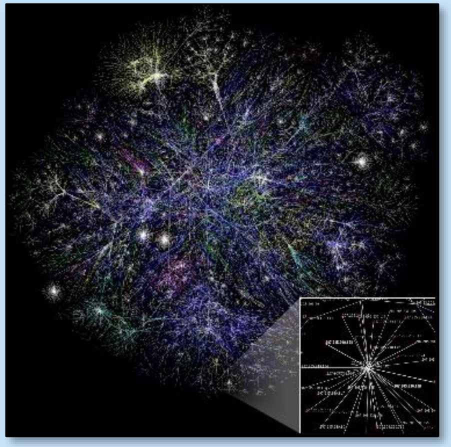

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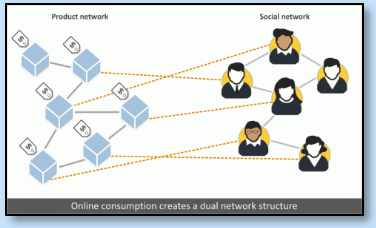

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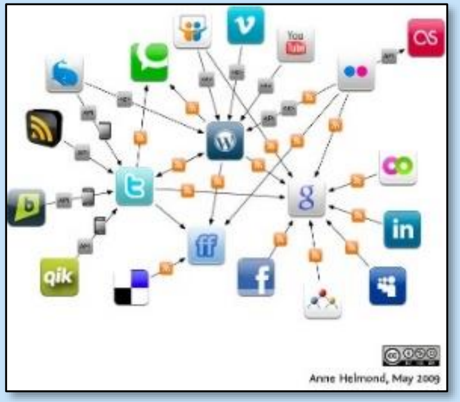

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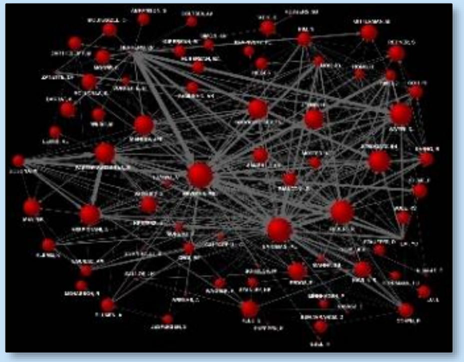
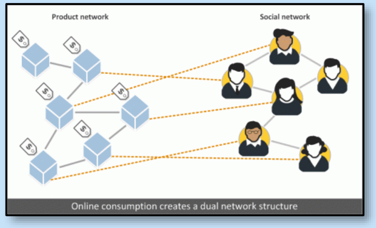
Citation & collaboration graphs





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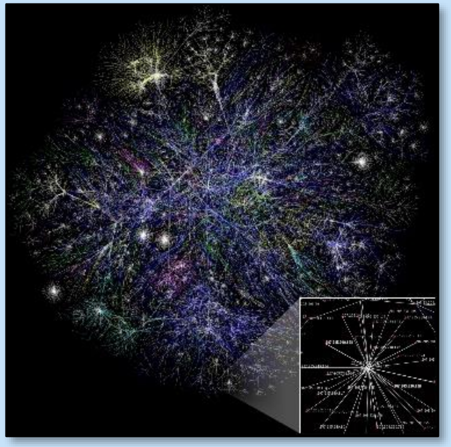


Web graphs

Machines

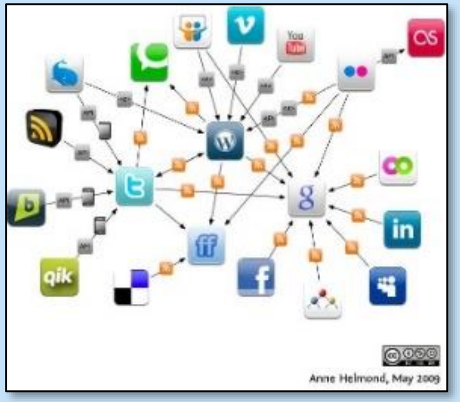
In-house Dell PowerEdge R910 server
(Intel Xeon X7550, 32 cores, 1TiB RAM)

Evaluation

Graphs

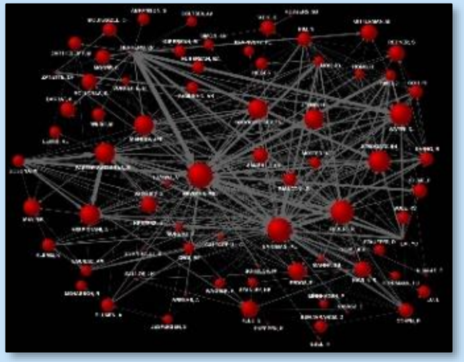
Road networks

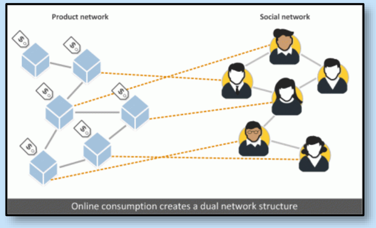
Social networks



Communication graphs

Citation & collaboration graphs





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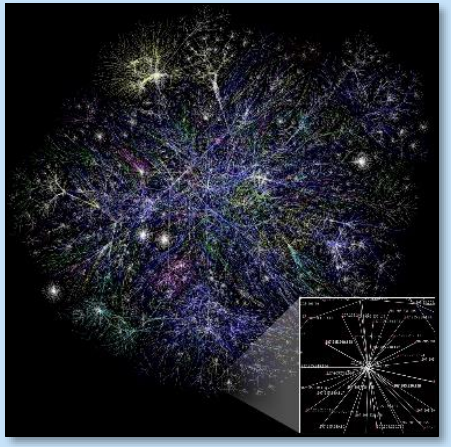


Web graphs

Machines

In-house Dell PowerEdge R910 server (Intel Xeon X7550, 32 cores, 1TiB RAM)

CSCS Ault, Intel Xeon Gold 6140, 18 cores, 768 GiB RAM

Evaluation

Evaluation

Comparison targets: 16 algorithms

Evaluation

Comparison targets: 16 algorithms

“First fit” (i.e., any order)

“Largest degree first”

“Smallest degree last”

Random

“Largest log-degree first”

“Smallest log-degree last”

(scheduling and speculative variants)

Evaluation

Comparison targets: 16 algorithms

“First fit” (i.e., any order)	Random
“Largest degree first”	“Largest log-degree first”
“Smallest degree last”	“Smallest log-degree last”

(scheduling and speculative variants)

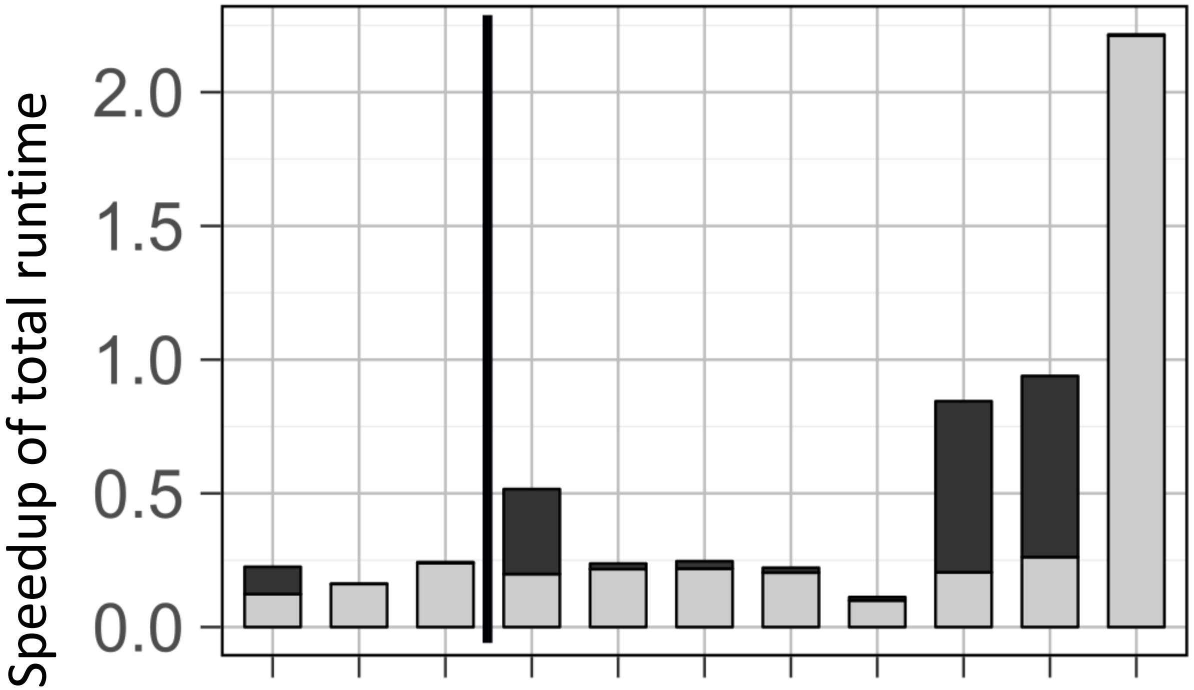
Taken from four libraries / codes

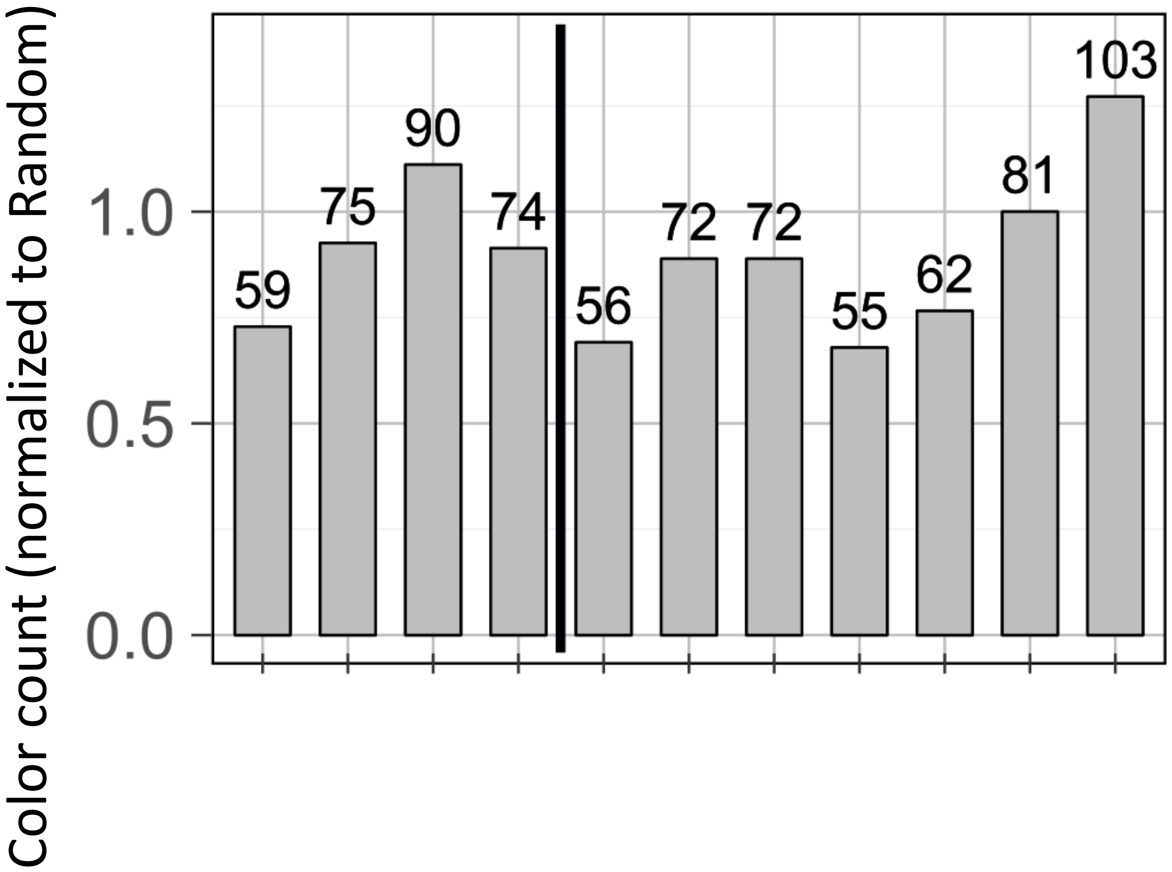
- Colpack [1]
- Zoltan [2]
- GBBS/Ligra [3]
- HP code [4]

[1] A. H. Gebremedhin, D. Nguyen, M. M. A. Patwary, and A. Pothen, “Colpack: Software for graph coloring and related problems in scientific computing”. TOMS’13.
[2] D. Bozdag, A. H. Gebremedhin, F. Manne, E. G. Boman, and U. V. ˘ Catalyurek, “A framework for scalable greedy coloring on distributedmemory parallel computers”. JPDC’08.
[3] L. Dhulipala, G. E. Blelloch, and J. Shun, “Theoretically efficient parallel graph algorithms can be fast and scalable” . SPAA’18.
[4] W. Hasenplaugh, T. Kaler, T. B. Schardl, and C. E. Leiserson, “Ordering heuristics for parallel graph coloring”. SPAA’14.

Runtime & quality analysis

Smaller graphs; 5M edges (used in online settings)





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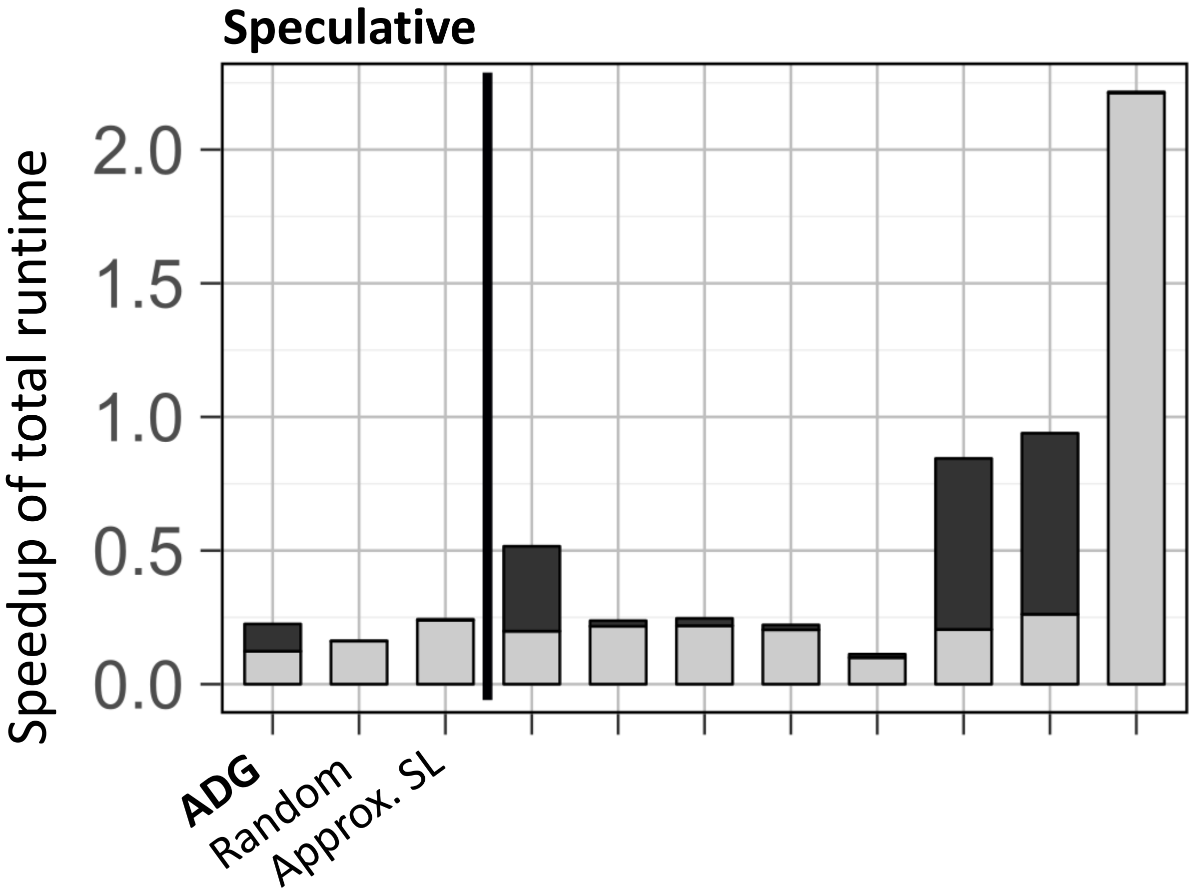
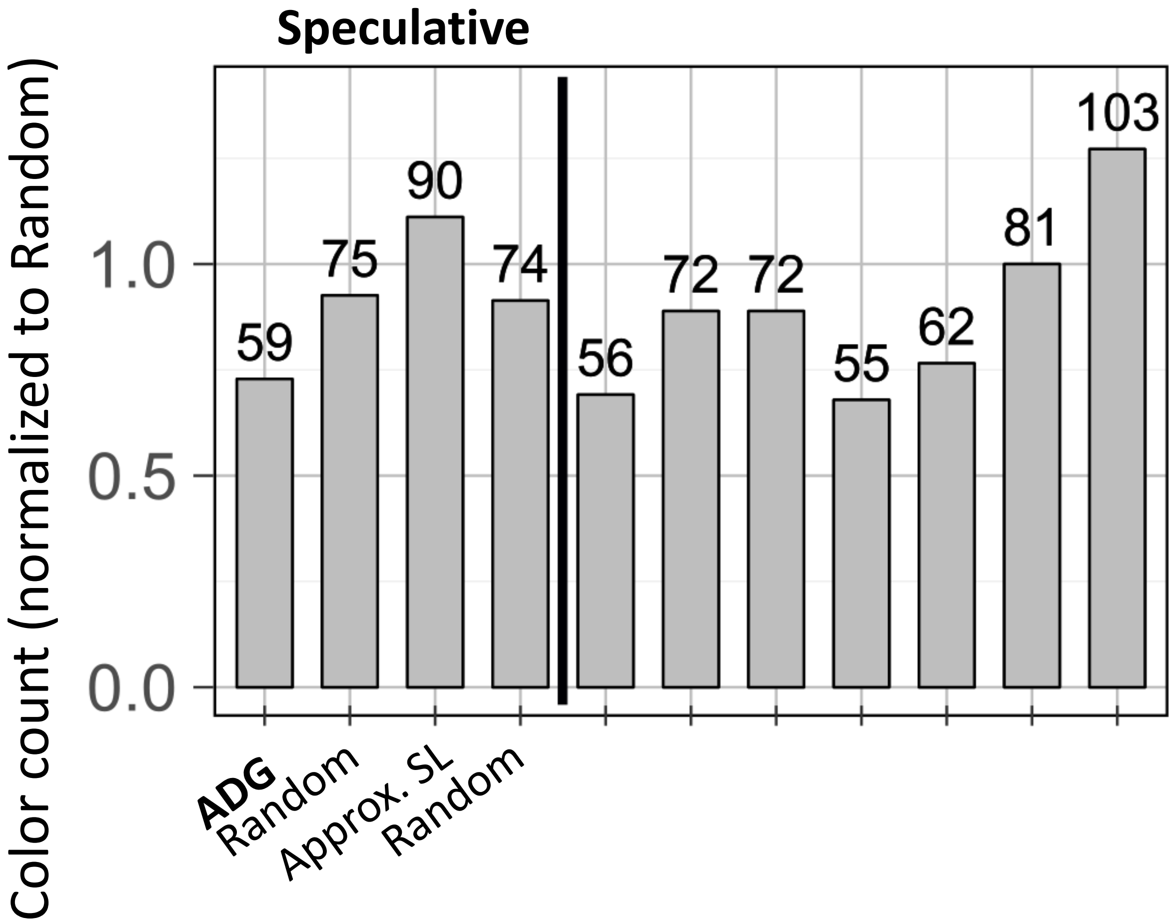

Speculative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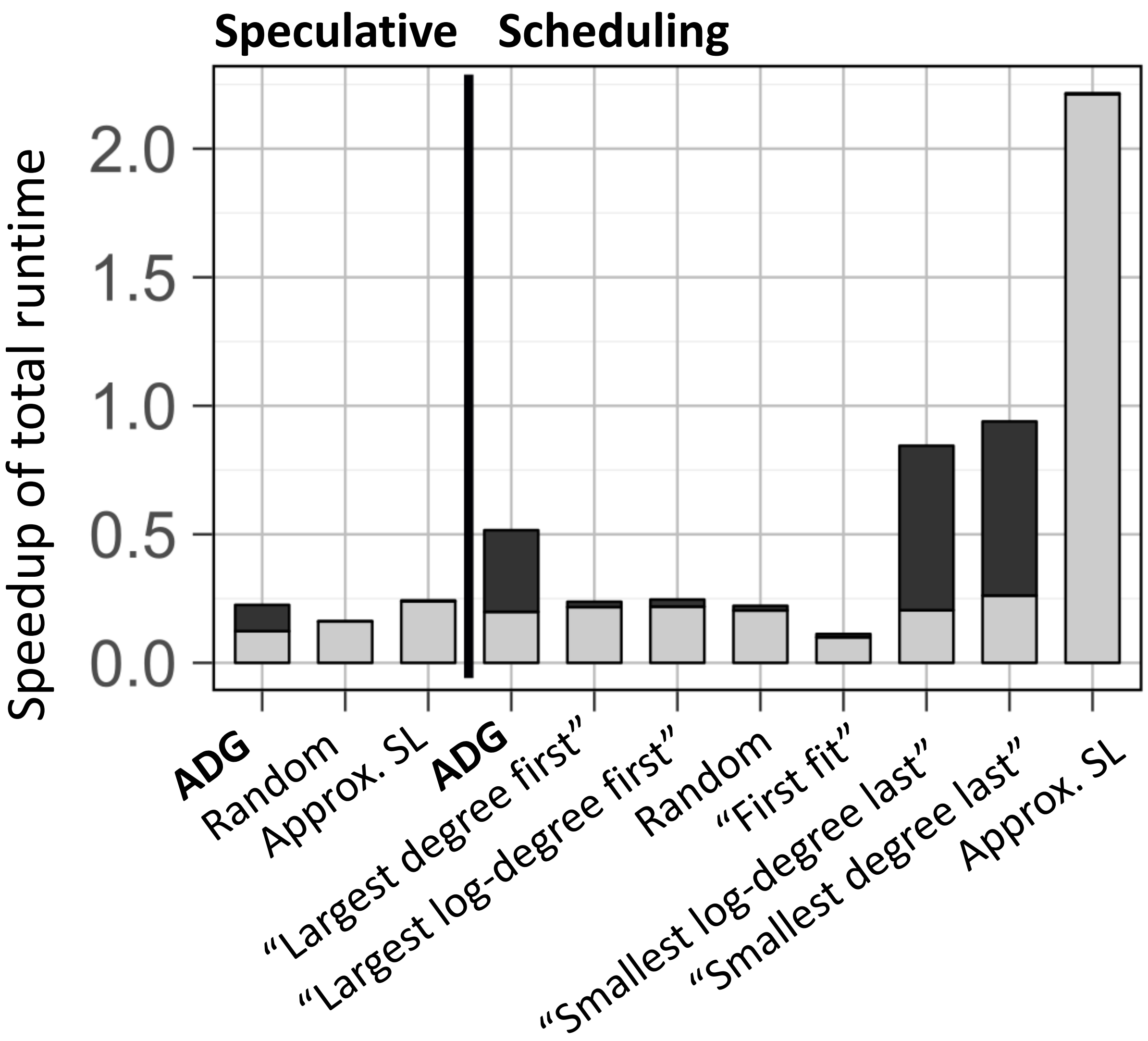

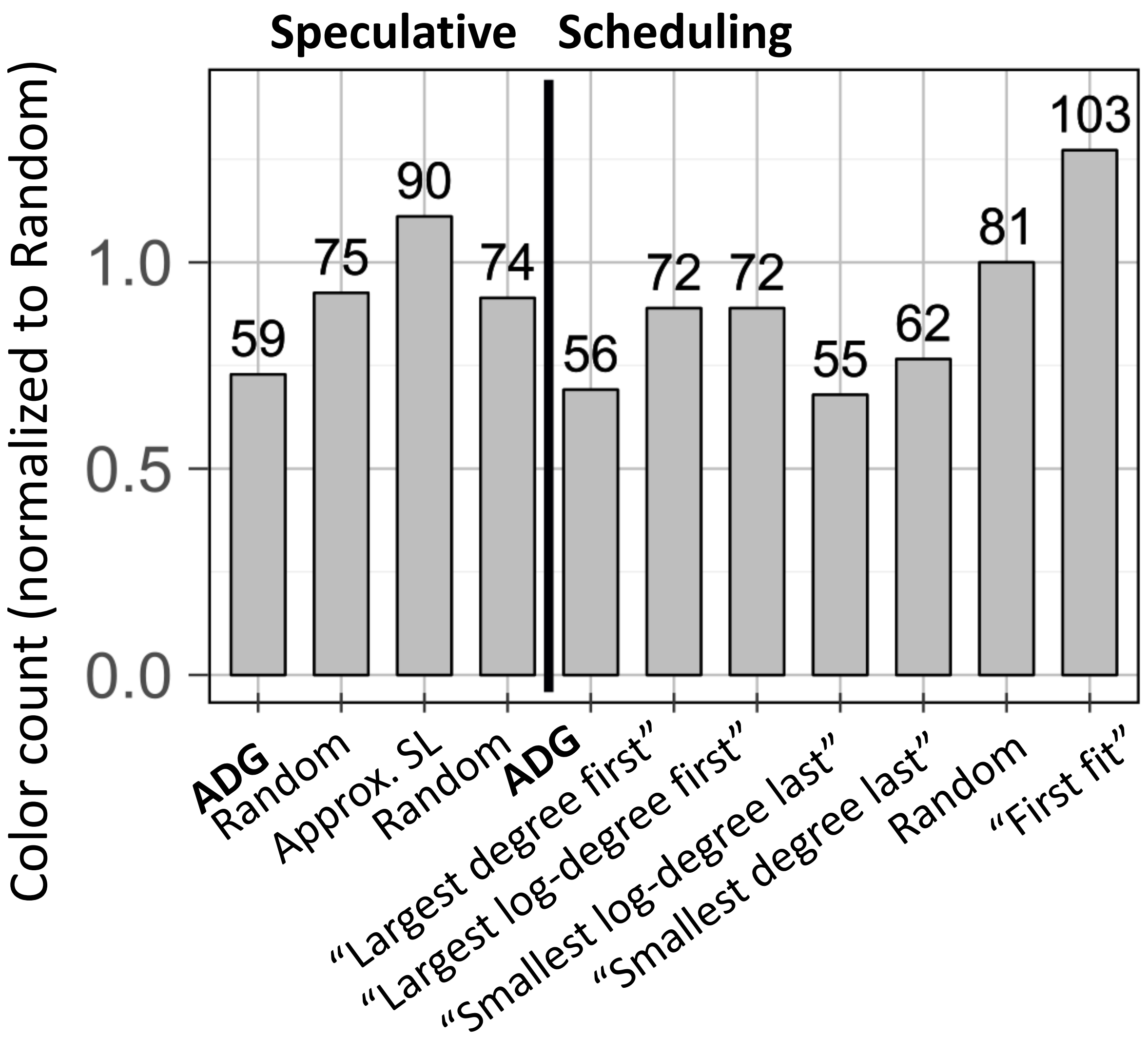

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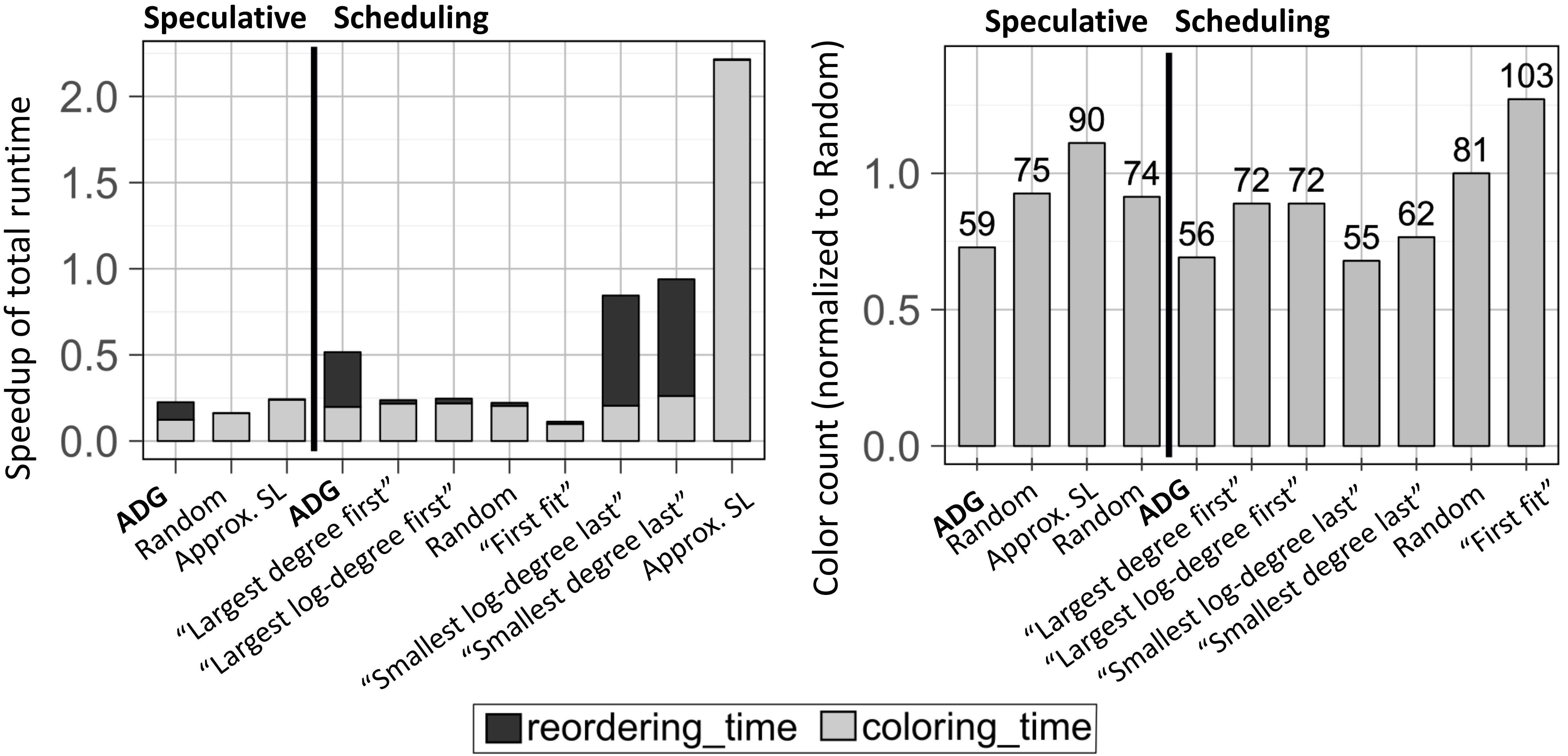
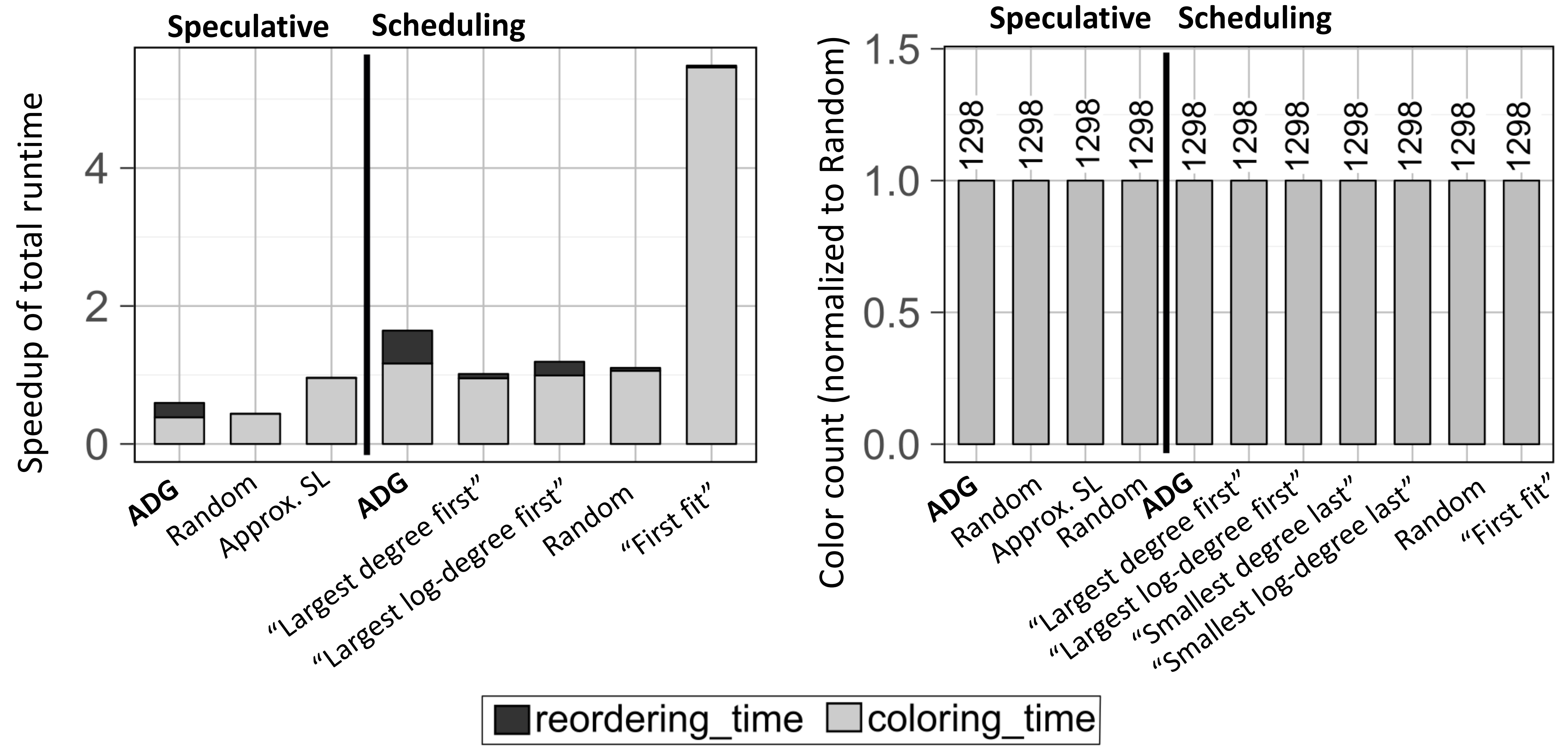


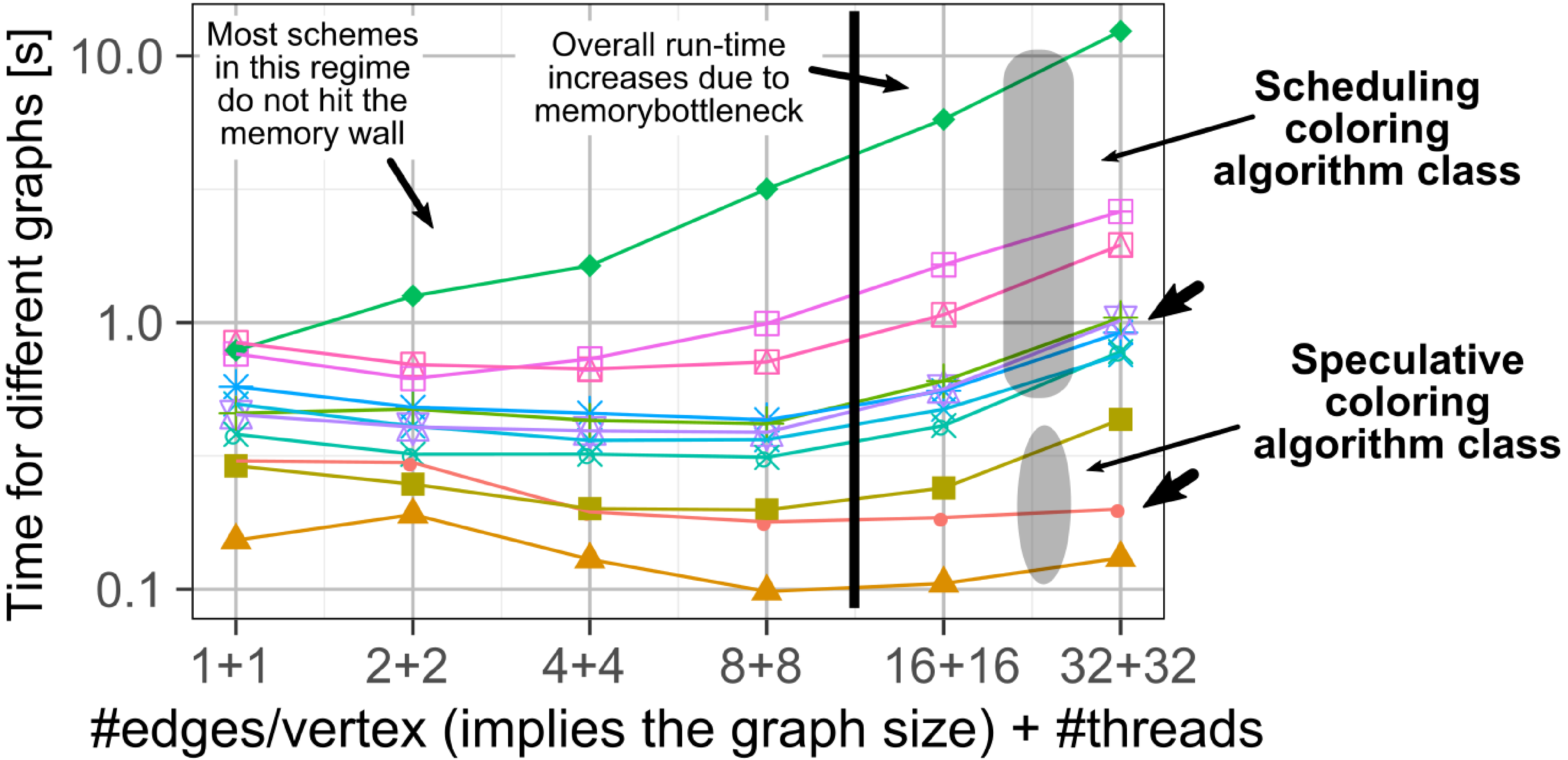
Runtime & quality analysis

Larger graphs, 230M edges (used in offline data analytics)



Scaling

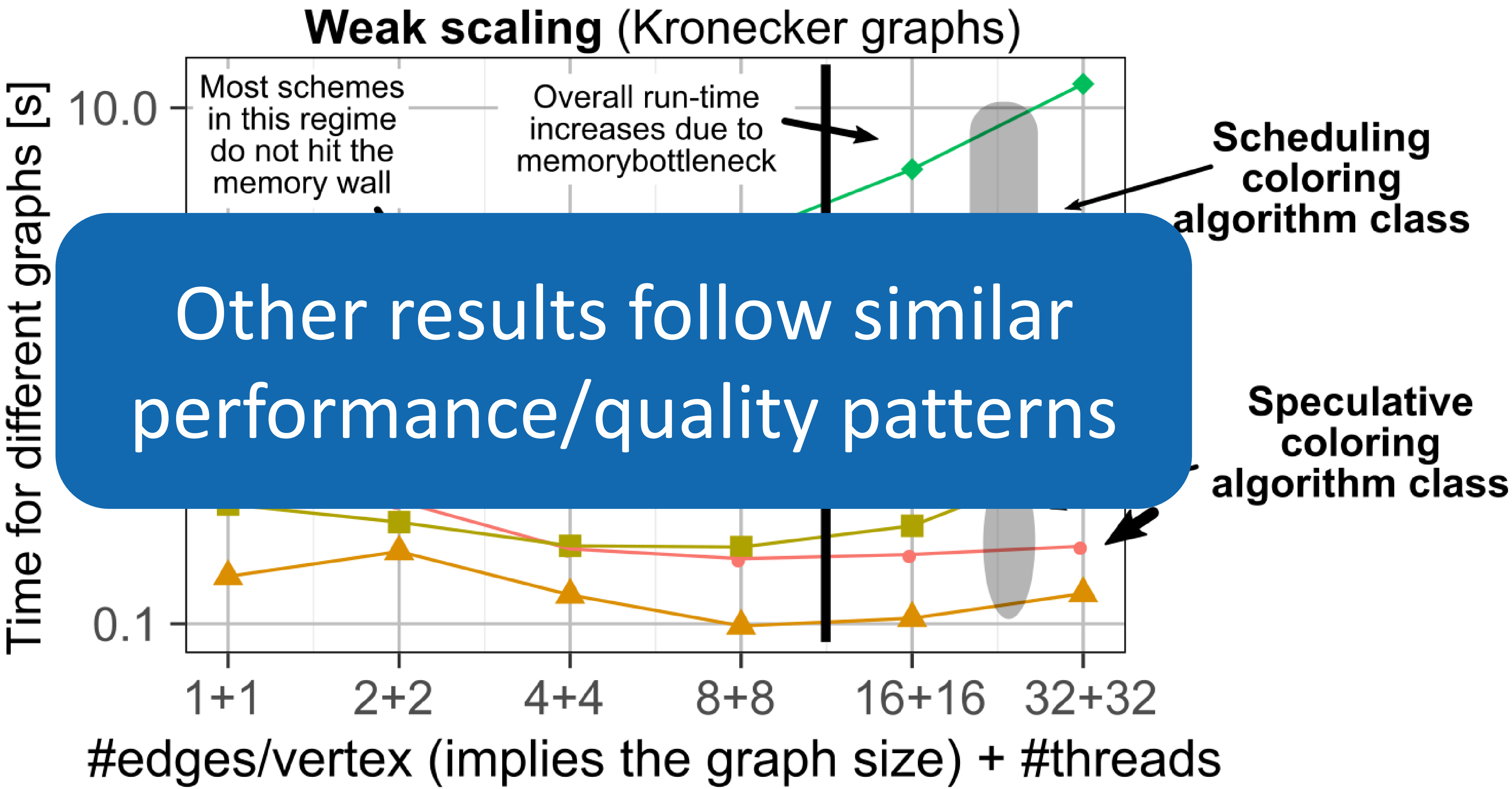
Weak scaling (Kronecker graphs)



Scheduling coloring algorithm class

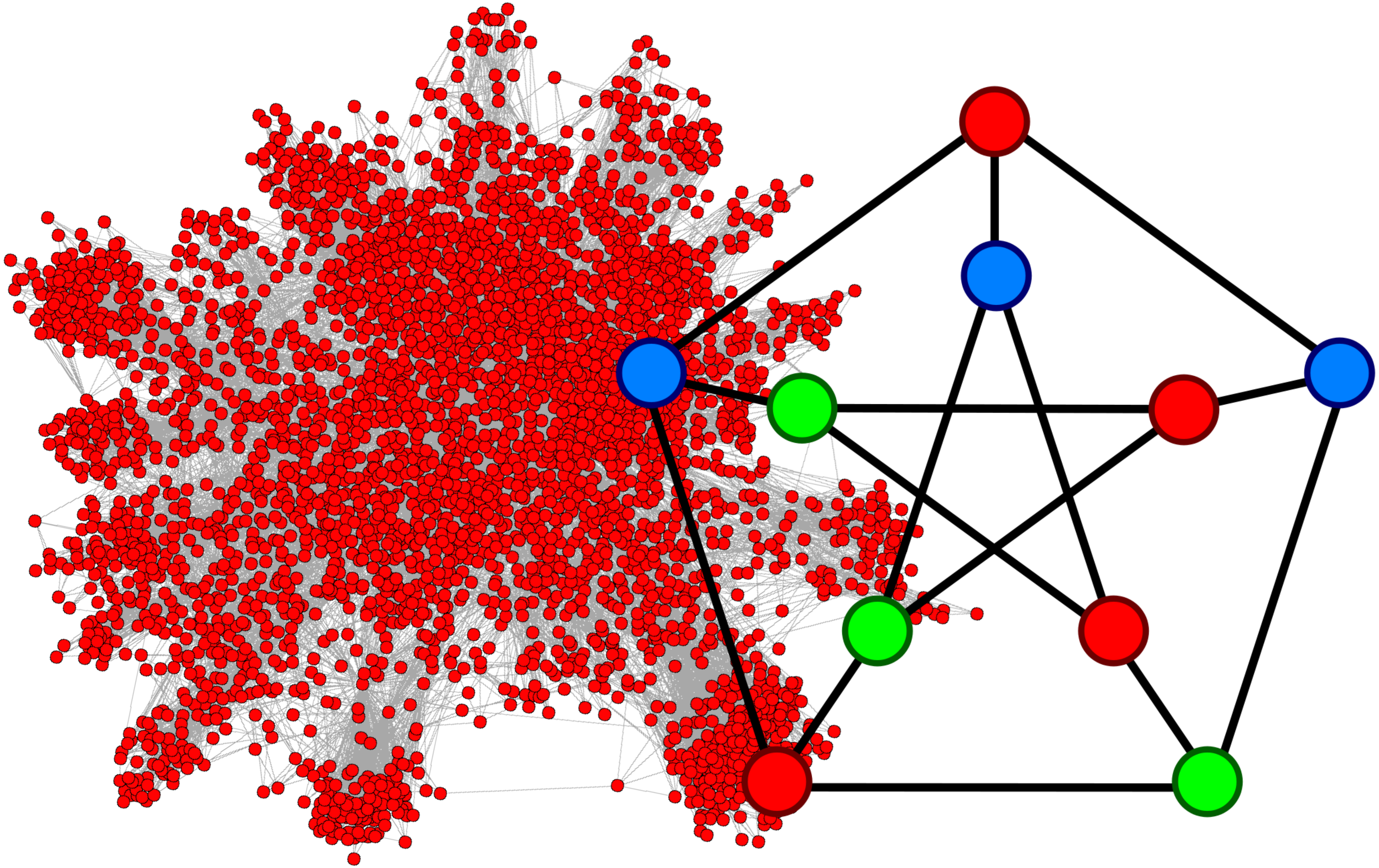
Speculative coloring algorithm class

Scaling



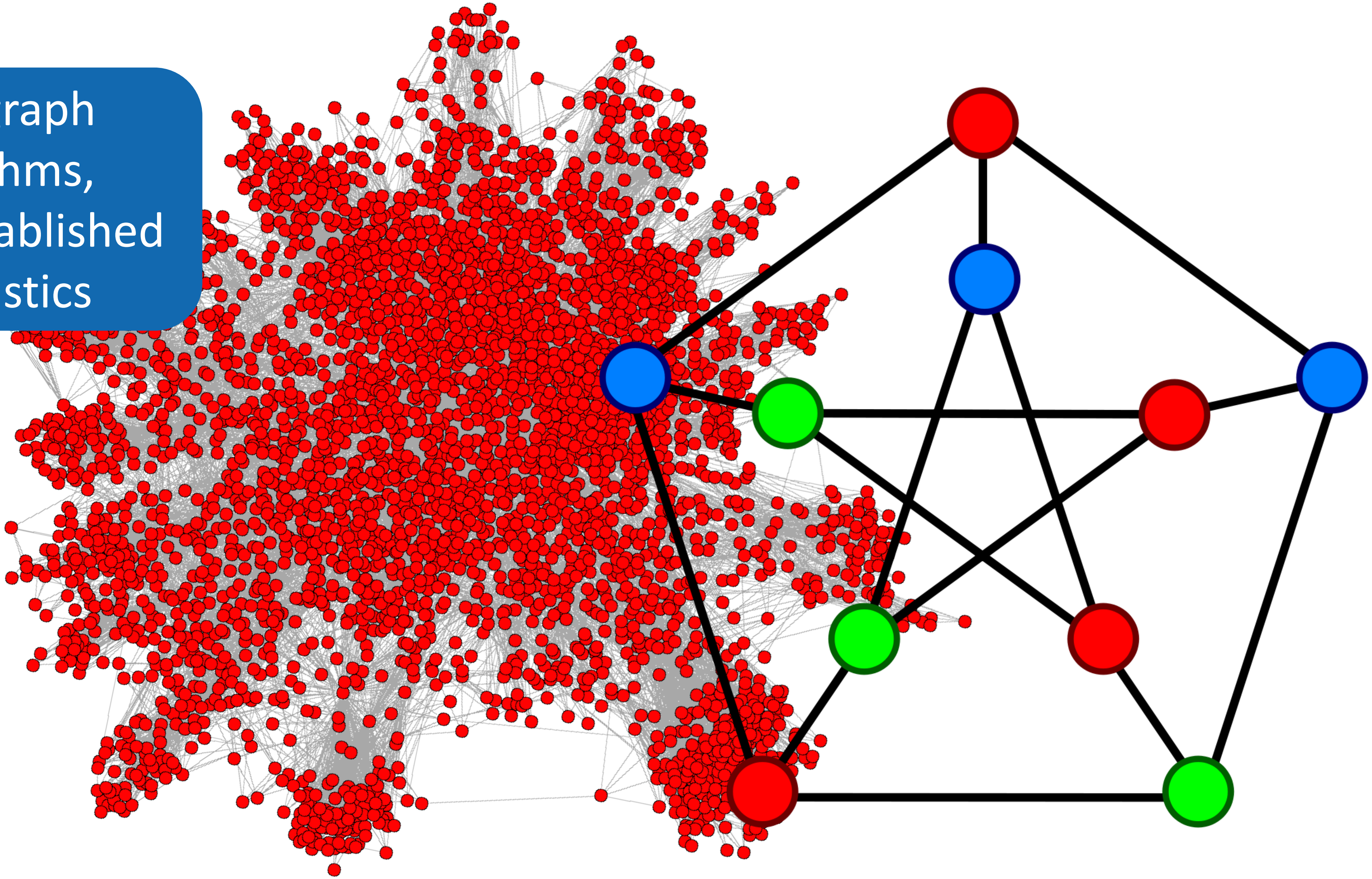
Other results follow similar performance/quality patterns

Conclusion



Conclusion

Novel parallel graph coloring algorithms, enhancing two established classes of heuristics



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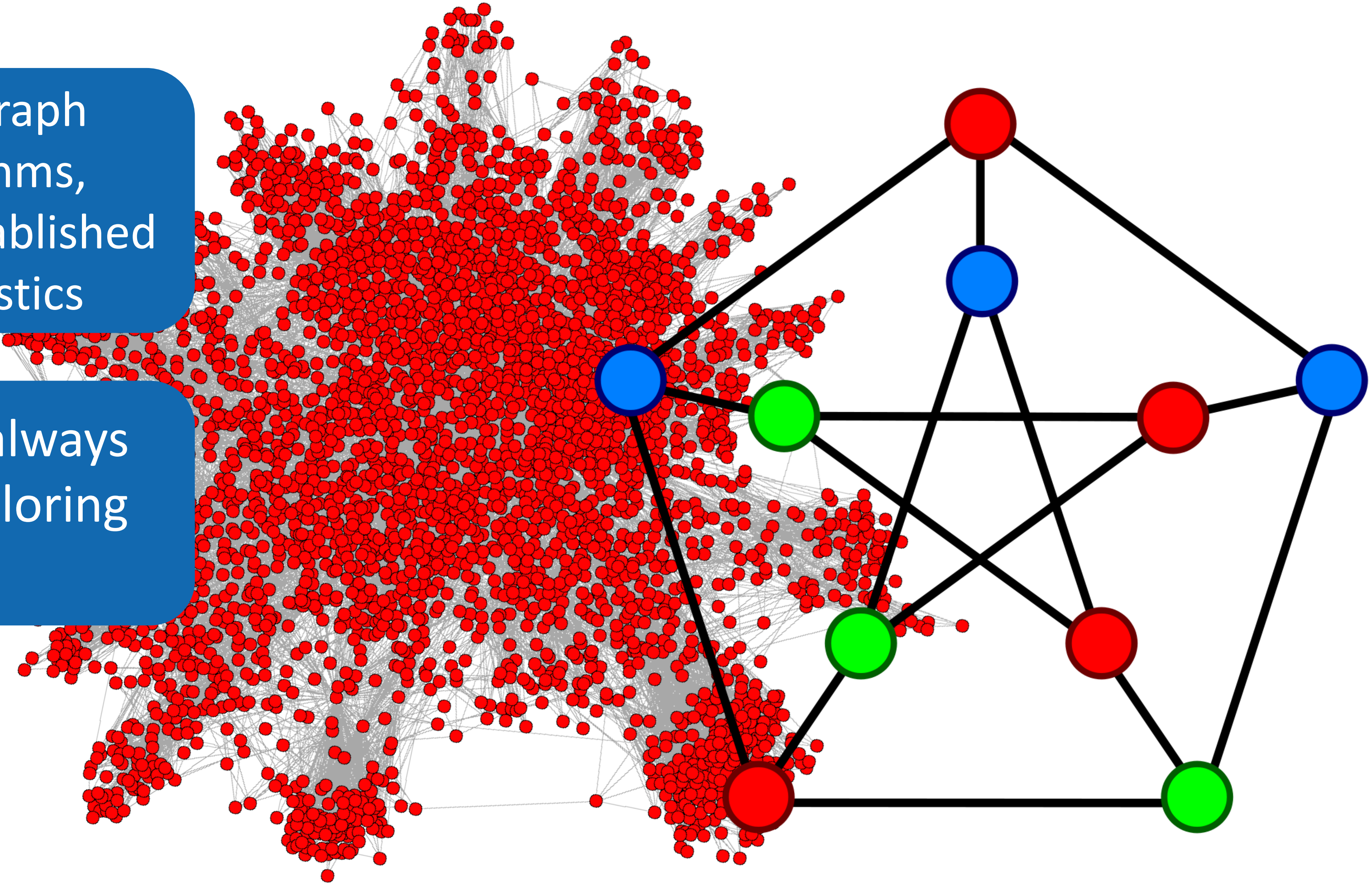

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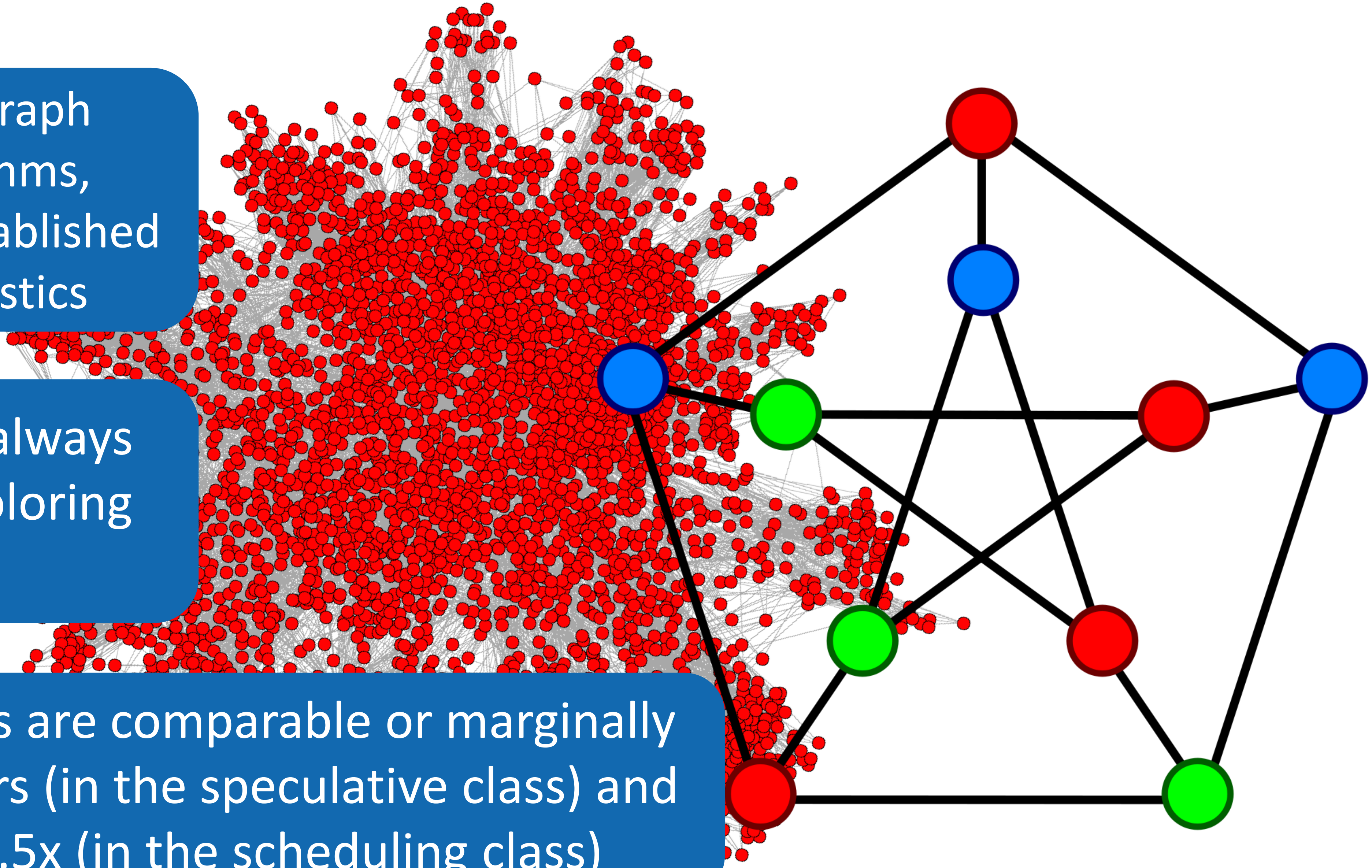

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→ Their runtimes are comparable or marginally higher than others (in the speculative class) and within 1.1 – 1.5x (in the scheduling class)

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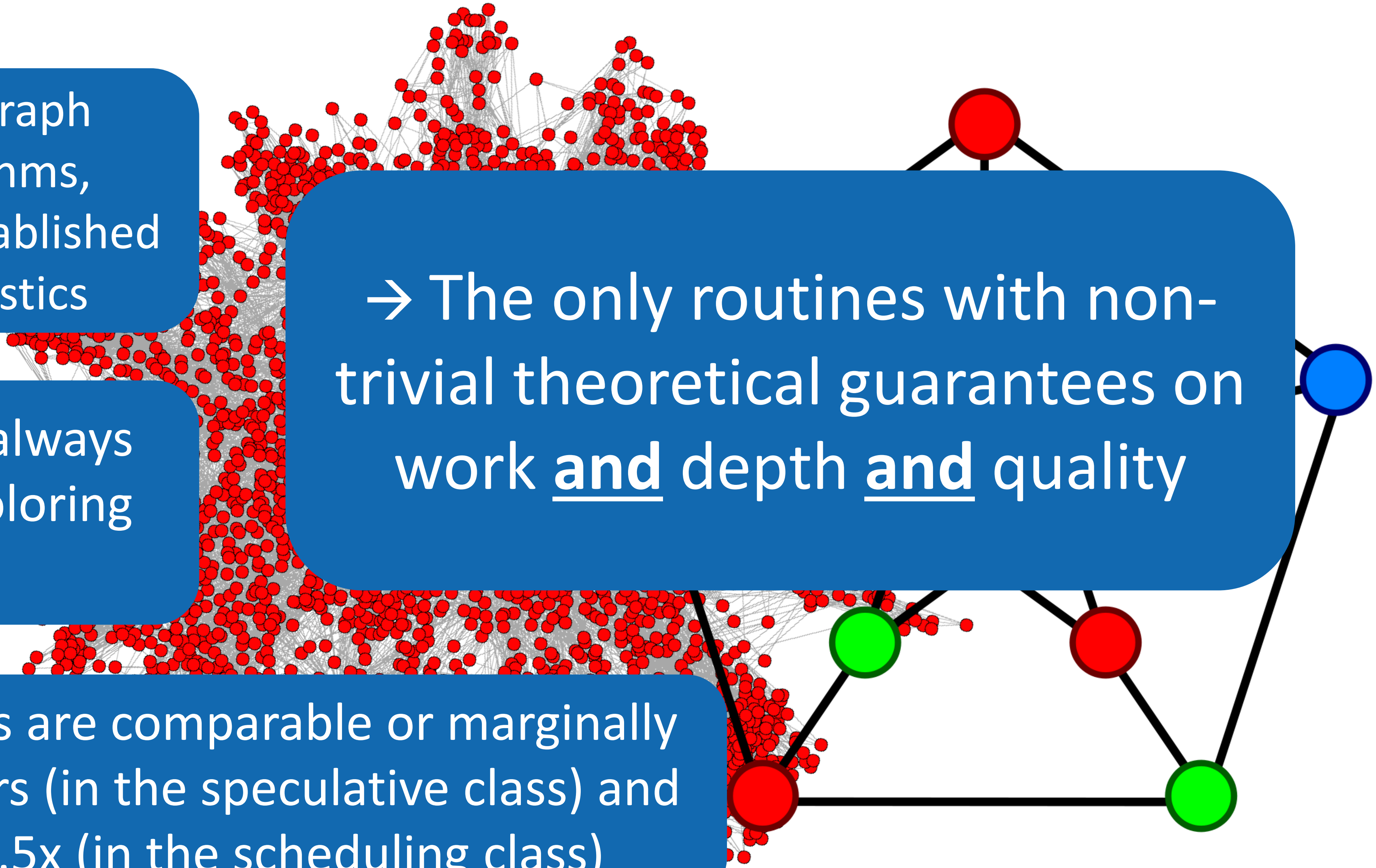


Novel parallel graph coloring algorithms, enhancing two established classes of heuristics

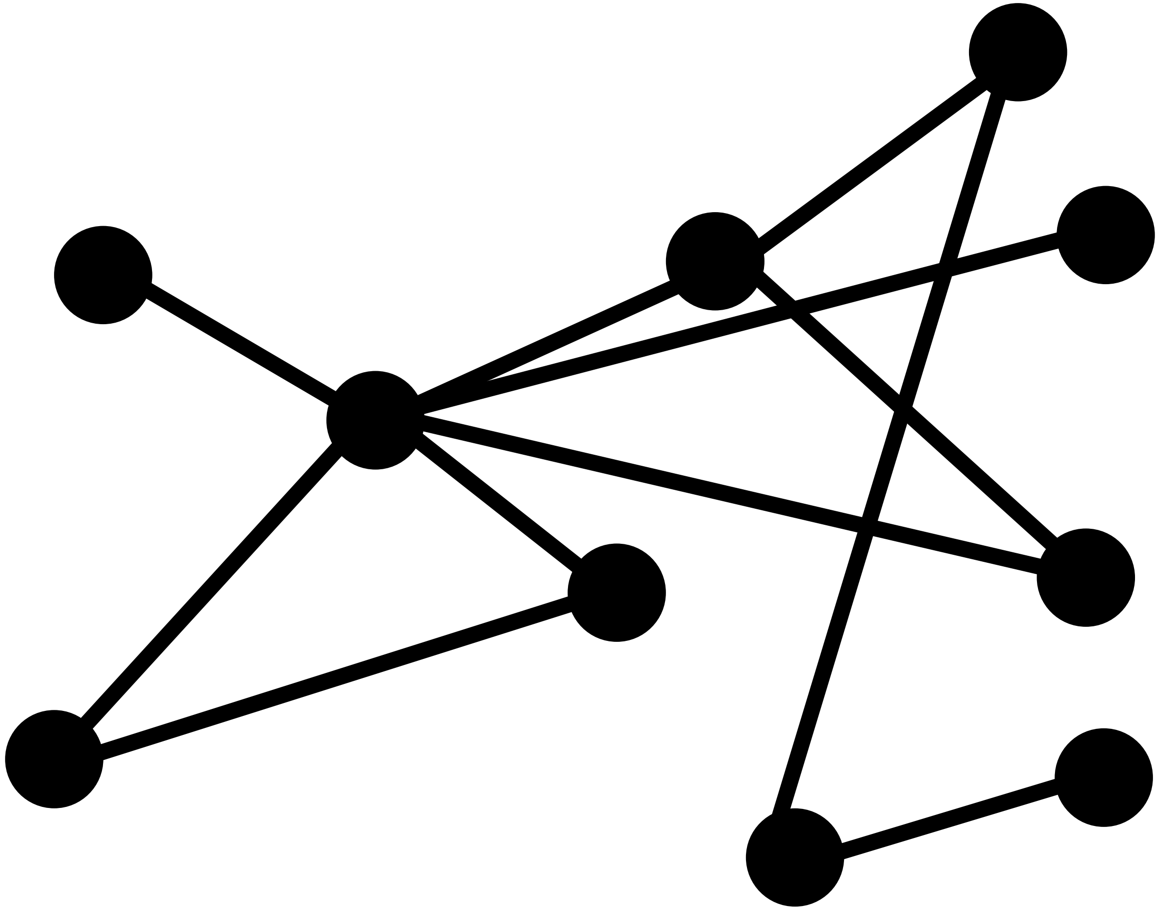
→ They almost always offer superior coloring quality

→ The only routines with non-trivial theoretical guarantees on work **and** depth **and** quality

→ Their runtimes are comparable or marginally higher than others (in the speculative class) and within 1.1 – 1.5x (in the scheduling class)

Backup Slides and Slides’ Variants

Graph degeneracy: an example



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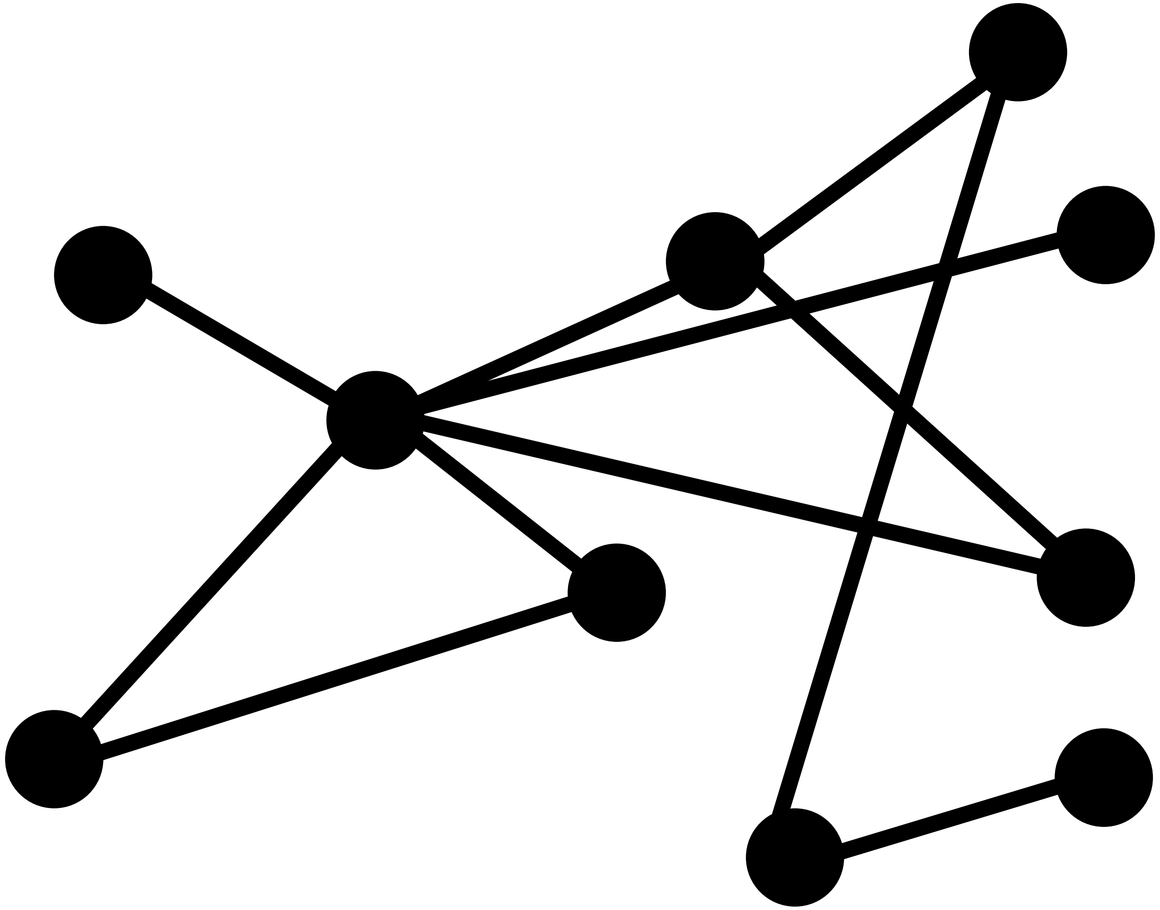

This graph has degeneracy of 3

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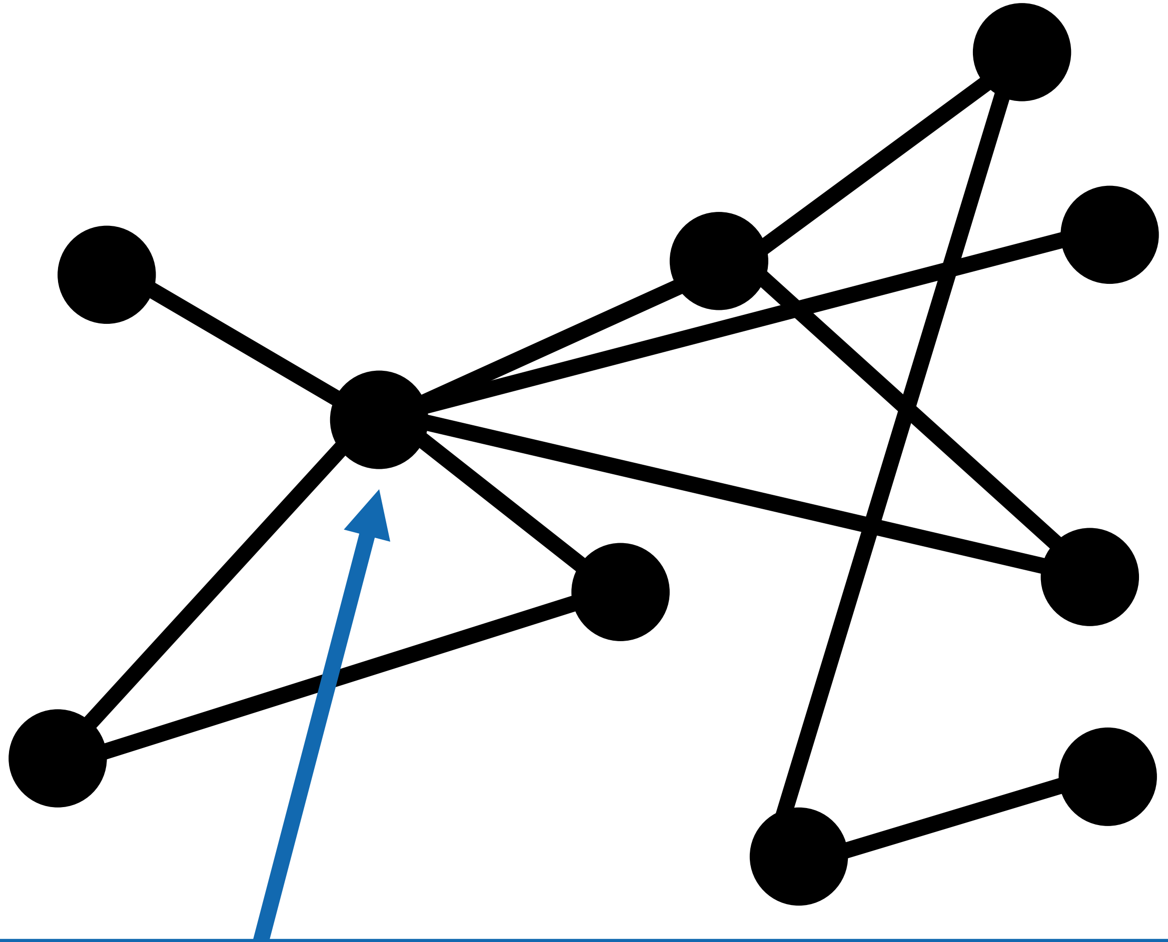

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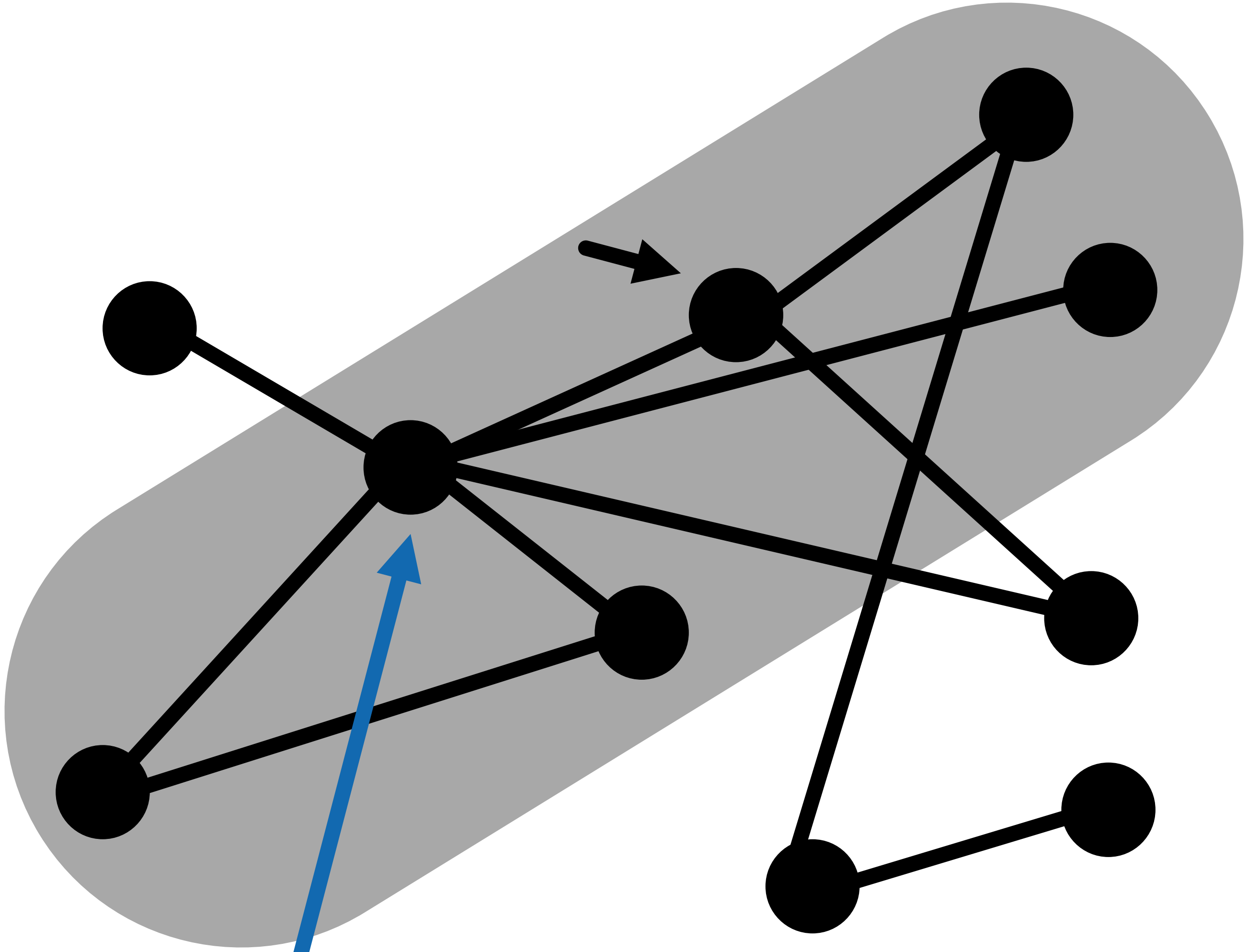

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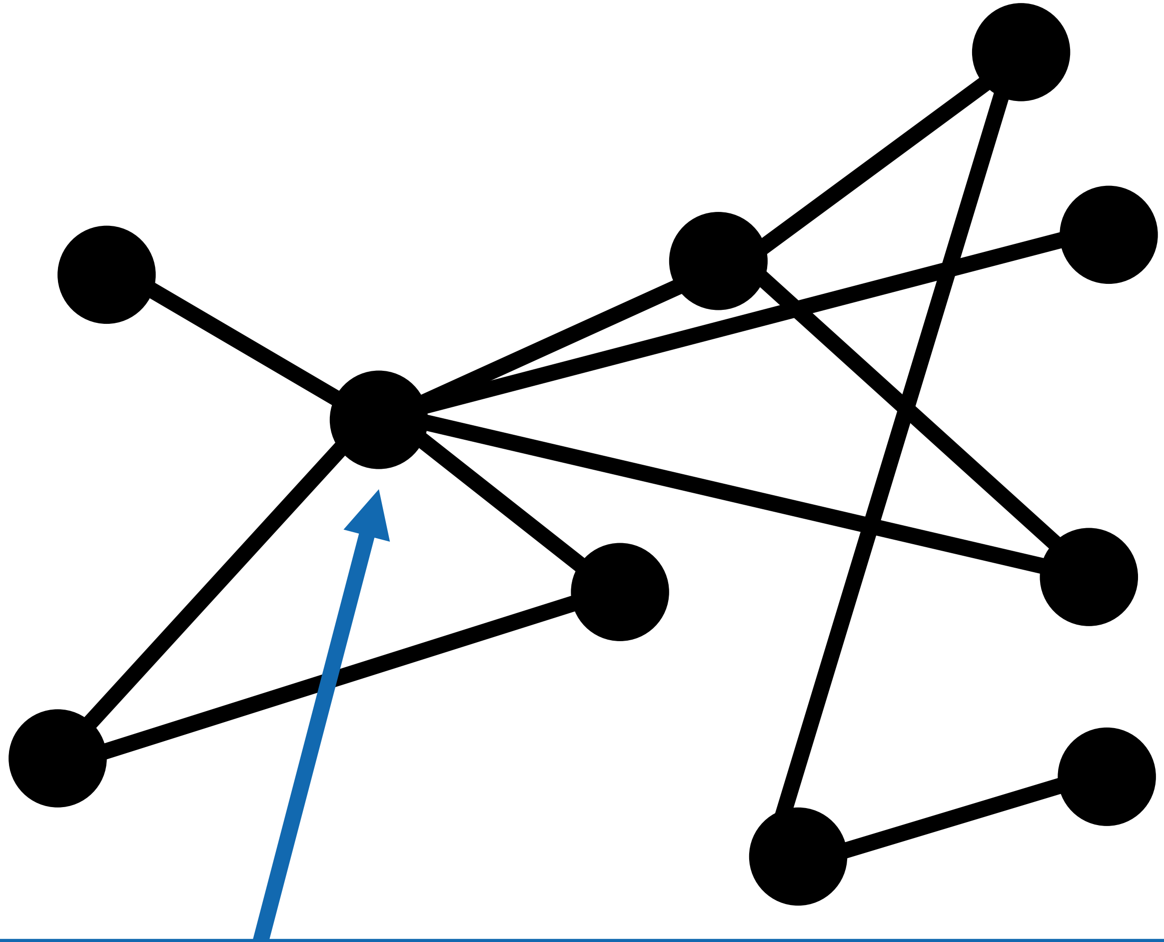

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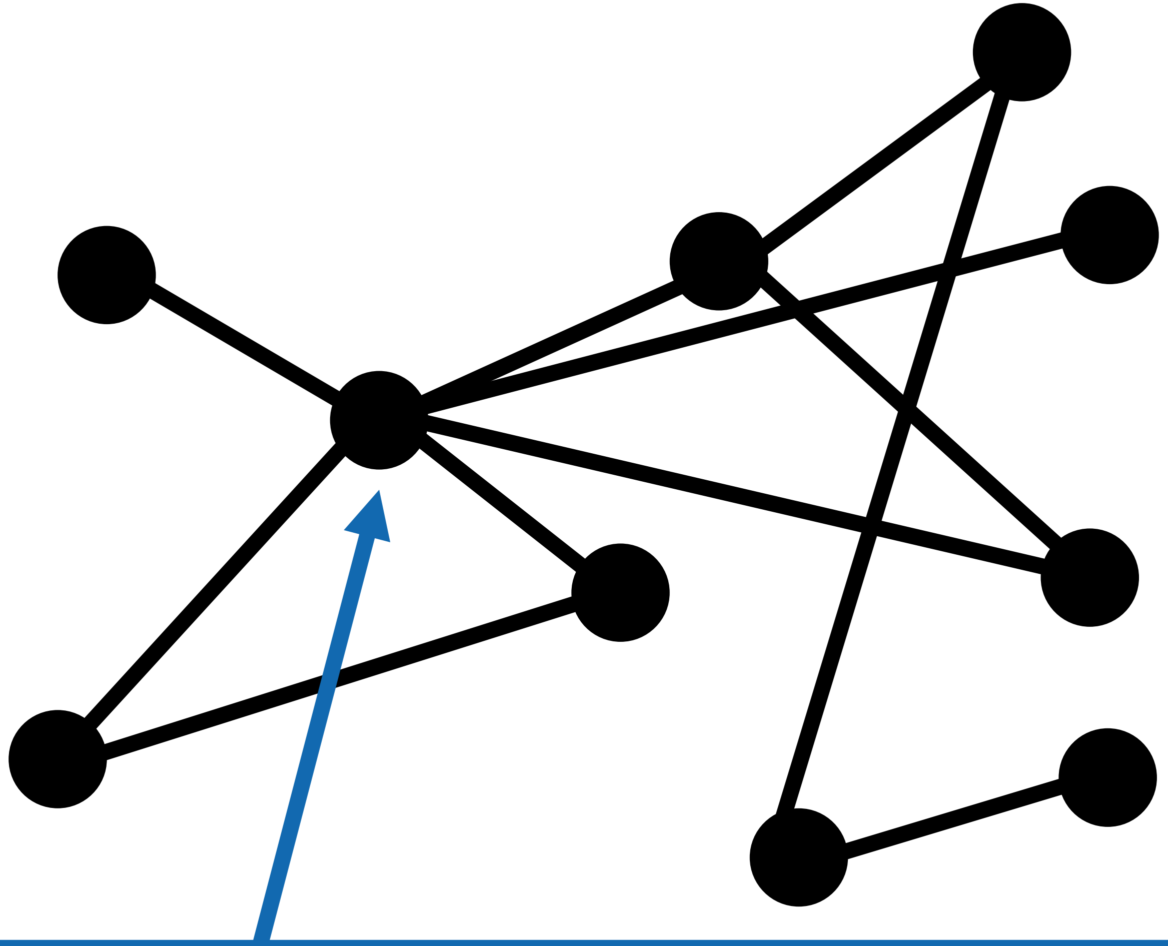

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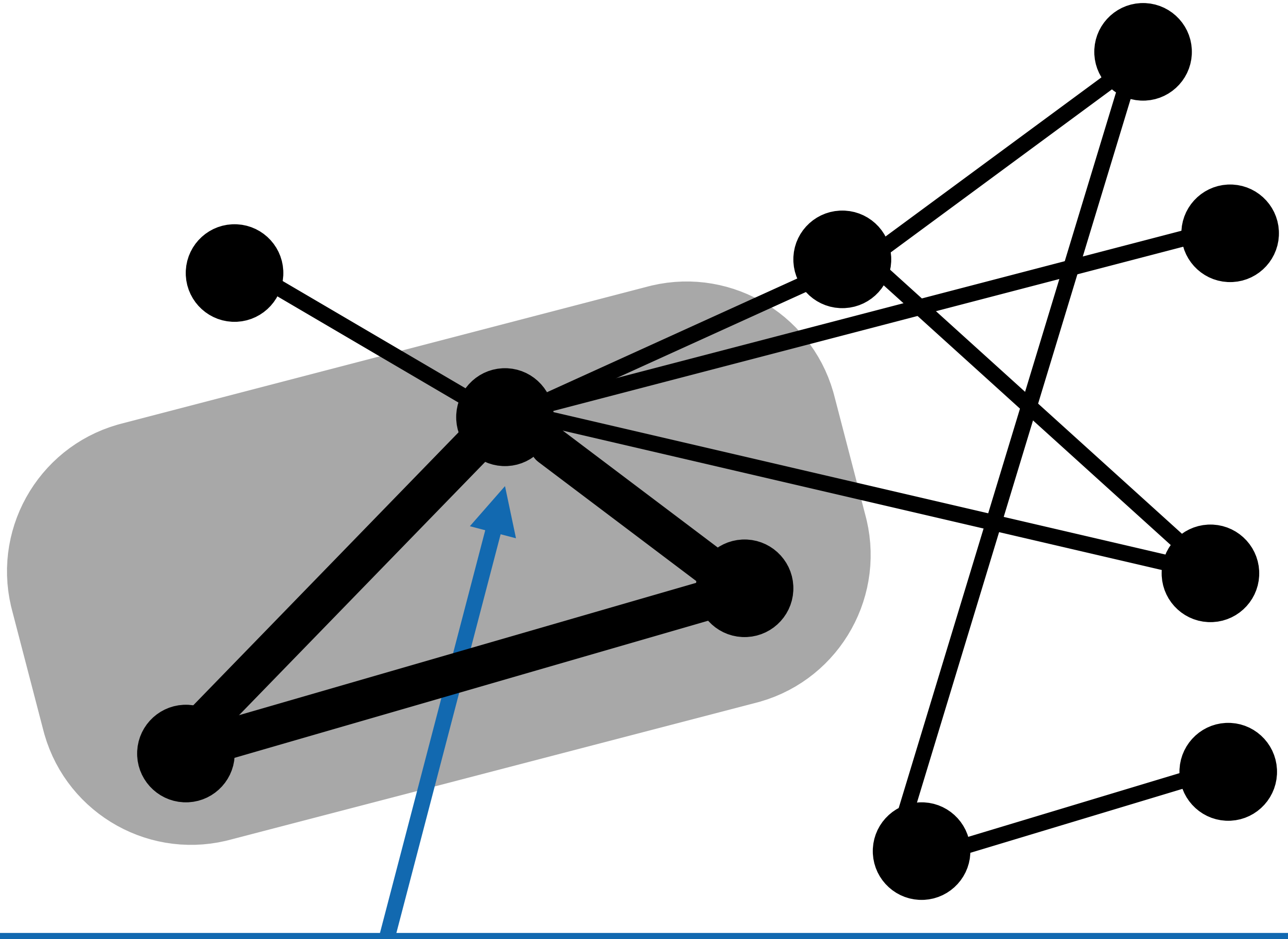

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“Smallest degree last”: fundamentals

→ Iterate over vertices in the **degeneracy ordering**

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/* n: number of vertices,
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? Can we have both good degeneracy-based quality and low depth & work ?

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Approximate degeneracy ordering

→ **Key idea**: try a relaxation of the strict degeneracy order, at the cost of (some) accuracy loss.

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/* n: the number of all vertices
   V: set of all vertices,
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One can prove that R_{min} forms a constant fraction of all vertices

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itr = 0;
while V ≠ Ø:
   $R_{min} = \{v \mid d(v) \le (1+\varepsilon)d_{avg}\}$;
   $V = V \setminus R_{min}$;
   forall v in $R_{min}$ in parallel:
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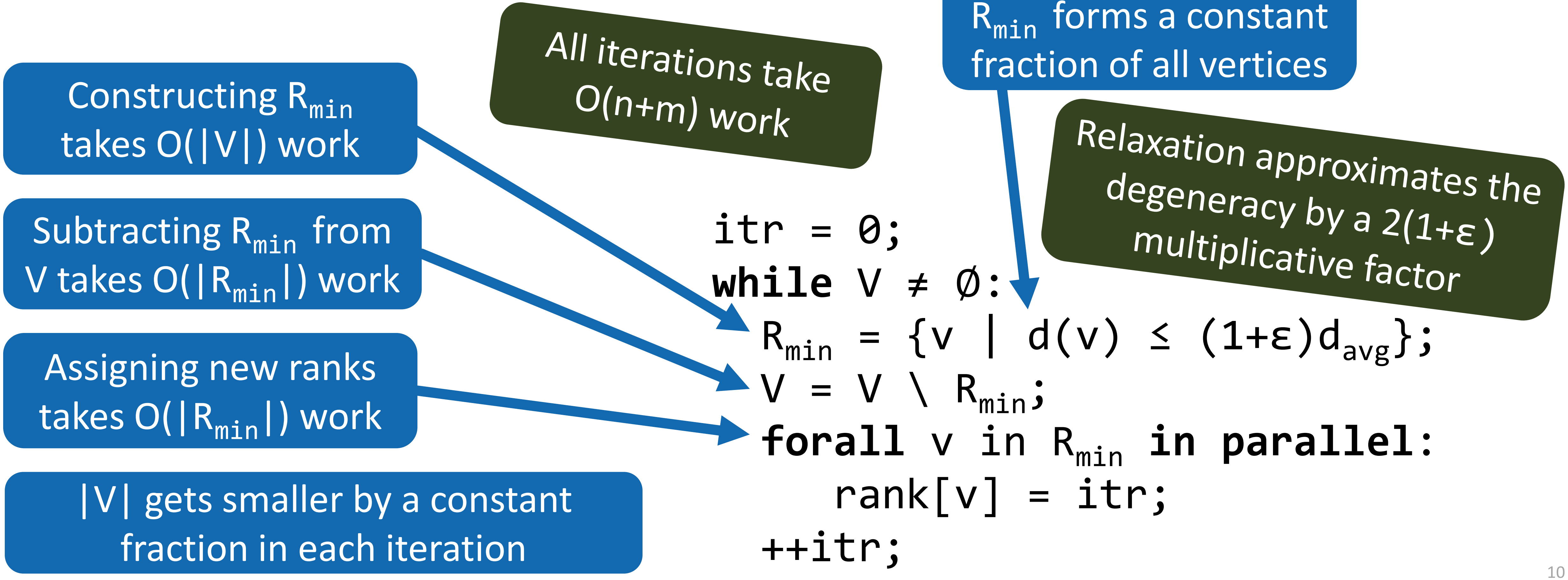

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Parallel graph coloring heuristics

Ordering	Depth	Work	Quality
“First fit” (i.e., any order)	No general bounds; $\Omega(n)$ for some graphs	$O(n+m)$	$\Delta+1$
“Largest degree first”	No general bounds; $\Omega(\Delta^2)$ for some graphs	$O(n+m)$	$\Delta+1$
“Smallest degree last”	No general bounds; $\Omega(n)$ for some graphs	$O(n+m)$	$d+1$
Random	$\mathbb{E}\, O\left(\frac{\log n}{\log\log n}\right)$	$O(n+m)$	$\Delta+1$
Random	$\mathbb{E}\, O\left(\log n + \log\Delta \cdot \min\left\{\sqrt{m}, \Delta + \frac{\log\Delta\log n}{\log\log n}\right\}\right)$	$O(n+m)$	$\Delta+1$
“Largest log-degree first”	$\mathbb{E}\, O\left(\log n + \log\Delta \cdot \left(\min\{\Delta, \sqrt{m}\} + \frac{\log^2\Delta\log n}{\log\log n}\right)\right)$	$O(n+m)$	$\Delta+1$
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ADG (scheduling)	$\mathbb{E}\, O\left(\log^2 n + \log\Delta \cdot \left(d\log n + \frac{\log d \cdot \log^2 n}{\log\log n}\right)\right)$	$O(n+m)$	$2(1+\varepsilon)d+1$

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All details, proofs, etc., are in the paper ☺

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→ **Key idea**: try a relaxation of the strict degeneracy order, at the cost of (some) accuracy loss.

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/* n: the number of all vertices
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itr = 0;
while $V \neq \emptyset$:
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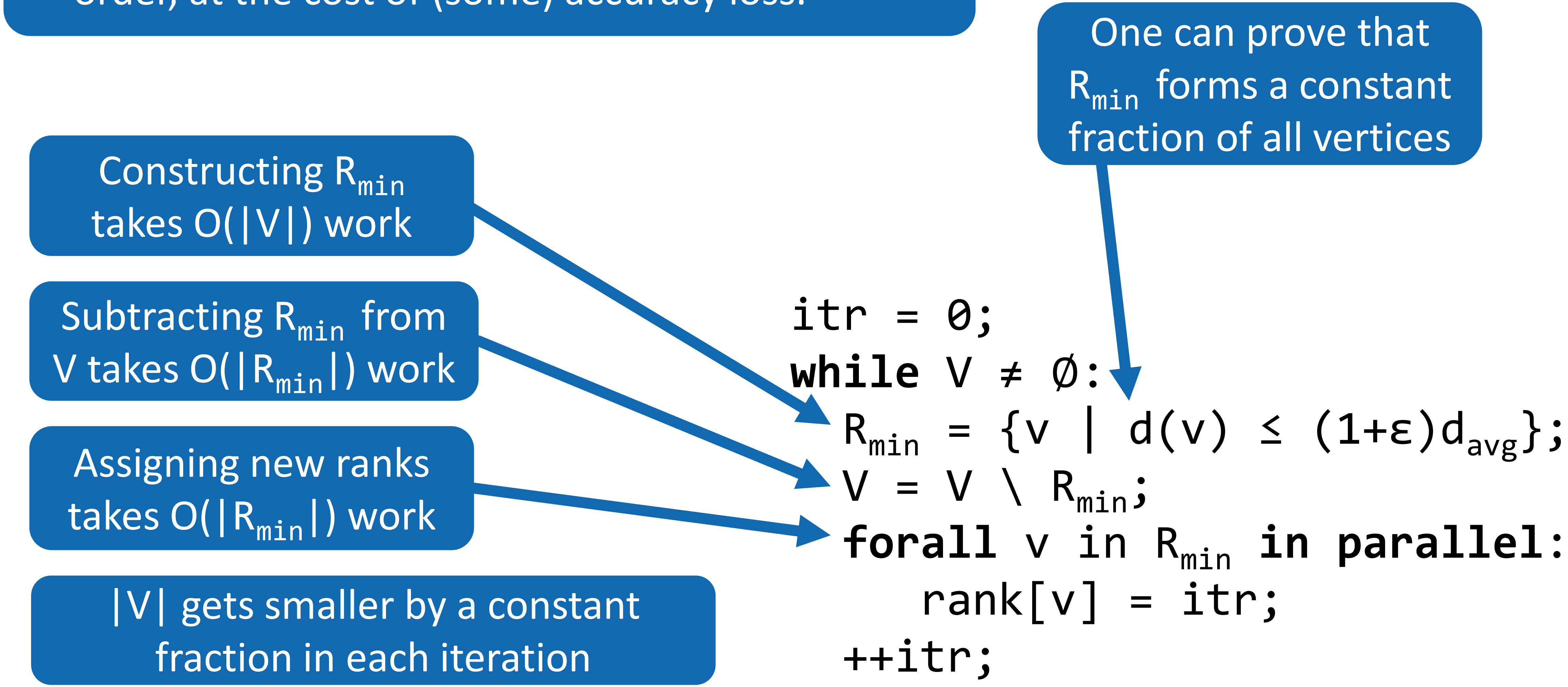

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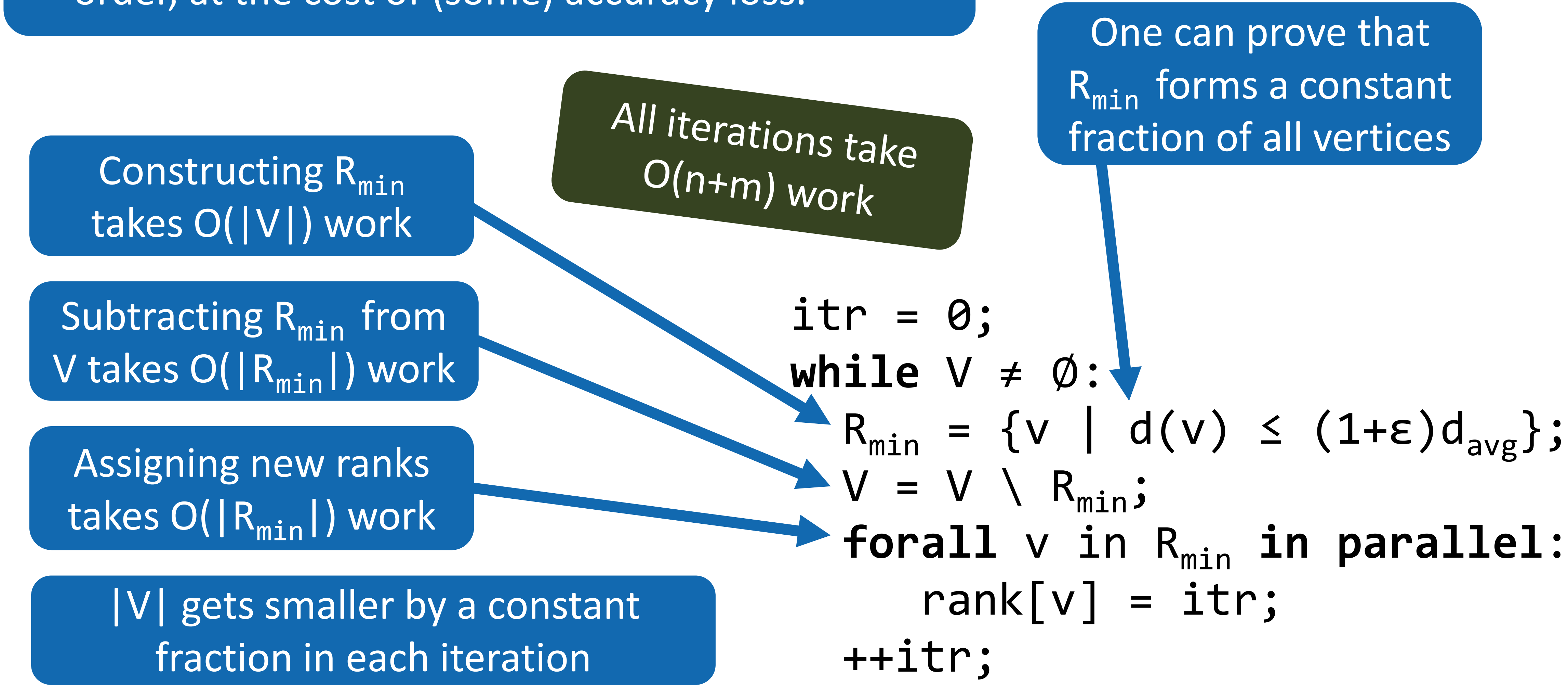

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Relaxation approximates the degeneracy by a 2(1+ε) multiplicative factor

Constructing R_{min} takes O(|V|) work

Subtracting R_{min} from V takes O(|R_{min}|) work

Assigning new ranks takes O(|R_{min}|) work

|V| gets smaller by a constant fraction in each iteration

```
itr = 0;
while V ≠ ∅:
    $R_{min}$ = {v | d(v) ≤ (1+ε)$d_{avg}$};
    V = V \ $R_{min}$;
    forall v in $R_{min}$ in parallel:
        rank[v] = itr;
    ++itr;
```